Volume 67B
Number 4
Oct.–Dec. 1963

# Mathematics and Mathematical Physics

U.S. DEPARTMENT OF COMMERCE

SECTION B

JOURNAL OF RESEARCH

NATIONAL BUREAU OF STANDARDS

# Journal of Research

of the

National Bureau of Standards

B. Mathematics and Mathematical Physics

OCT.–DEC. • 1963

VOLUME • 67B

NUMBER • 4

Editor: Chester H. Page

Associate Editors: William H. Pell, John J. Sopka

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1963


## JOURNAL OF RESEARCH

The National Bureau of Standards Journal of Research reports research and development in the fields of activity shown at right. Also included from time to time are survey articles on topics closely related to the Bureau's scientific and technical program.

The Journal is published in four separate sections as follows:

**A. Physics and Chemistry**

Contains papers of interest primarily to scientists working in these fields. Issued six times a year. Annual subscription: domestic, $4.00; foreign, $4.75.

**B. Mathematics and Mathematical Physics**

Presents studies and compilations designed mainly for the mathematician and the theoretical physicist. Issued quarterly. Annual subscription: domestic, $2.25; foreign, $2.75.

**C. Engineering and Instrumentation**

Reports research and development results of interest chiefly to the engineer and the applied scientist. Issued quarterly. Annual subscription: domestic, $2.25; foreign, $2.75.

**D. Radio Propagation**

Reports work in radio propagation, upper atmospheric physics, and communications. Issued six times a year. Annual subscription: domestic, $4.00; foreign, $4.75.


U.S. DEPARTMENT OF COMMERCE
Luther H. Hodges, Secretary
NATIONAL BUREAU OF STANDARDS
A. V. Astin, Director


**Functions and Activities**

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to government agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services, and various consultation and information services. The Bureau also serves as the Federal technical research center in a number of specialized fields. The scope of activities of the National Bureau of Standards is suggested in the following listing of the divisions and sections engaged in technical work.

**Washington, D.C.**

**Electricity.** Resistance and Reactance. Electrochemistry. Electrical Instruments. Magnetic Measurements. Dielectrics. High Voltage. Absolute Electrical Measurements.

**Metrology.** Photometry and Colorimetry. Refractometry. Photographic Research. Length. Engineering Metrology. Mass and Volume.

**Heat.** Temperature Physics. Heat Measurements. Cryogenic Physics. Equation of State. Statistical Physics.

**Radiation Physics.** X-ray. Radioactivity. Radiation Theory. High Energy Radiation. Radiological Equipment. Nucleonic Instrumentation. Neutron Physics.

**Analytical and Inorganic Chemistry.** Pure Substances. Spectrochemistry. Solution Chemistry. Standard Reference Materials. Applied Analytical Research. Crystal Chemistry.

**Mechanics.** Sound. Pressure and Vacuum. Fluid Mechanics. Engineering Mechanics. Rheology. Combustion Controls.

**Polymers.** Macromolecules: Synthesis and Structure. Polymer Chemistry. Polymer Physics. Polymer Characterization. Polymer Evaluation and Testing. Applied Polymer Standards and Research. Dental Research.

**Metallurgy.** Engineering Metallurgy. Metal Reactions. Metal Physics. Electrolysis and Metal Deposition.

**Inorganic Solids.** Engineering Ceramics. Glass. Solid State Chemistry. Crystal Growth. Physical Properties. Crystallography.

**Building Research.** Structural Engineering. Fire Research. Mechanical Systems. Organic Building Materials. Codes and Safety Standards. Heat Transfer. Inorganic Building Materials. Metallic Building Materials.

**Applied Mathematics.** Numerical Analysis. Computation. Statistical Engineering. Mathematical Physics. Operations Research.

**Data Processing Systems.** Components and Techniques. Computer Technology. Measurements Automation. Engineering Applications. Systems Analysis.

**Atomic Physics.** Spectroscopy. Infrared Spectroscopy. Far Ultraviolet Physics. Solid State Physics. Electron Physics. Atomic Physics. Plasma Spectroscopy.

**Instrumentation.** Engineering Electronics. Electron Devices. Electronic Instrumentation. Mechanical Instruments. Basic Instrumentation.

**Physical Chemistry.** Thermochemistry. Surface Chemistry. Organic Chemistry. Molecular Spectroscopy. Elementary Processes. Mass Spectrometry. Photochemistry and Radiation Chemistry.

**Office of Weights and Measures.**

**Boulder, Colo.**

**Cryogenic Engineering Laboratory.** Cryogenic Processes. Cryogenic Properties of Solids. Cryogenic Technical Services. Properties of Cryogenic Fluids.

### CENTRAL RADIO PROPAGATION LABORATORY

**Ionosphere Research and Propagation.** Low Frequency and Very Low Frequency Research. Ionosphere Research. Prediction. Services. Sun-Earth Relationships. Field Engineering. Radio Warning Services. Vertical Soundings Research.

**Troposphere and Space Telecommunications.** Data Reduction Instrumentation. Radio Noise. Tropospheric Measurements. Tropospheric Analysis. Spectrum Utilization Research. Radio-Meteorology. Lower Atmosphere Physics.

**Radio Systems.** Applied Electromagnetic Theory. High Frequency and Very High Frequency Research. Frequency Utilization. Modulation Research. Antenna Research. Radiodetermination.

**Upper Atmosphere and Space Physics.** Upper Atmosphere and Plasma Physics. High Latitude Ionosphere Physics. Ionosphere and Exosphere Scatter. Airglow and Aurora. Ionospheric Radio Astronomy.

### RADIO STANDARDS LABORATORY

**Radio Standards Physics.** Frequency and Time Disseminations. Radio and Microwave Materials. Atomic Frequency and Time-Interval Standards. Radio Plasma. Microwave Physics.

**Radio Standards Engineering.** High Frequency Electrical Standards. High Frequency Calibration Services. High Frequency Impedance Standards. Microwave Calibration Services. Microwave Circuit Standards. Low Frequency Calibration Services.

**JOINT INSTITUTE FOR LABORATORY ASTROPHYSICS—NBS GROUP (UNIV. OF COLO.).**


Order all publications from the Superintendent of Documents
U.S. Government Printing Office, Washington, D.C., 20402.

Use of funds for printing this publication approved by the Director of the Bureau of the Budget (June 19, 1961).

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 4, October–December 1963

# Zonal Harmonic Perturbations of an Accurate Reference Orbit of an Artificial Satellite*

John P. Vinti

(July 24, 1963)

The theory developed in an earlier paper, for an accurate reference orbit of an artificial satellite, is first slightly modified, so as to prepare the way for a treatment of zonal harmonic perturbations. Delaunay variables are next introduced, by means of certain linear combinations of the action variables, along with their canonical conjugates. Application of the von Zeipel method then permits the calculation of the most important zonal harmonic perturbations. These arise from the third, with coefficient $J_3$, and the residual fourth, with coefficient $J_4+J_2^2$. The accuracy of the secular and short-periodic effects is through terms of order $J_2^2$ and that of the long-periodic effects is through terms of order $J_2$. Since the reference orbit itself, with its exact secular terms, takes care of all but 0.5 percent of the deviation of the earth's gravitational field from spherical symmetry, the overall secular accuracy of the final orbit surpasses that of other second order theories. The results are compared with those of Kozai.

## 1. Reference Orbit

The author [Vinti 1959a,b] has introduced a potential

$$V' = -\mu\rho(\rho^2 + c^2\eta^2)^{-1} \tag{1.00}$$

that can represent accurately the gravitational field of an oblate planet. Here $\mu$ is the product of the gravitational constant $G$ and the planet's mass, $c$ is an adjustable length, and $\rho$ and $\eta$ are oblate spheroidal coordinates, defined by the equations

$$X + iY = r\cos\theta \exp i\phi = [(\rho^2 + c^2)(1-\eta^2)]^{1/2} \exp i\phi \tag{1.01}$$

$$Z = r\sin\theta = \rho\eta. \tag{1.02}$$

If an artificial satellite is at the field point, $r$, $\theta$, and $\phi$ are respectively its planetocentric distance, declination, and right ascension, and $X$, $Y$, and $Z$ are its rectangular coordinates, $OZ$ being along the planet's axis and $OX$ pointing toward its vernal equinox.

If $r_e$ is the equatorial radius, the true potential is

$$V = -\mu r^{-1}[1 - \sum_{n=2}^{\infty}(r_e/r)^n J_n P_n(\sin\theta)] + \text{tesseral harmonics}, \tag{1.03}$$

where only the products $r_e^n J_n$ need to be known. That is, differences in the definition of $r_e$, when noncircularity of the equator is taken into account, can be reconciled by small adjustments of the $J$'s. Then if

$$c = r_e J_2^{1/2}, \tag{1.04}$$

*Research supported by the National Aeronautics and Space Agency, U.S.A.

$V'$ represents exactly the zeroth harmonic $-\mu/r$ and the second harmonic and also gives higher even harmonics, characterized by

$$J_{2m}=(-1)^{m+1}J_2^m. \tag{1.05}$$

In particular it gives $J_4+J_2^2=0$, as compared with observed values for the earth ranging from $-(0.9)10^{-6}$ to $(0.4)10^{-6}$ (Kaula 1962; King-Hele, Cook, and Rees 1963). Consequently it accounts for about 99.5 percent of the deviation of $V$ from the value $-\mu/r$ corresponding to spherical symmetry. It thus accounts almost completely for the flattening of the earth, leading to a geoid that never departs by more than about 30 m from the true sea-level surface.

For the drag-free motion of an artificial satellite the potential (1.00) leads to a separable problem, which has been worked out analytically [Vinti 1961 a,b, 1962]. This solution, holding for all angles of inclination and containing no critical inclination or long-periodic terms, gives secular terms exactly by means of rapidly converging infinite series and short-periodic terms correctly through order $J_2^2$. We call this orbit corresponding to (1.00) the reference orbit. For such a reference orbit error can never accumulate, because of the exactness of the secular terms, and the periodic terms can be in error only by amounts of the order $J_2^3$, i.e., by about 1 part in $10^9$, since $J_2=(1.08)10^{-3}$ for the earth.

## 2. Zonal Harmonic Perturbations

For a satellite of the earth, if its orbit is high enough so that drag is small and low enough so that the moon's effect is small, the above reference orbit ought to hold rather well for a good many revolutions. (I purposely choose vague words here, since numerical comparisons are still incomplete.) Eventually, however, the actual orbit will deviate more and more from such a reference orbit, because of the neglected forces. These include forces arising from drag, meteoritic impact, radiation, electromagnetic fields, the sun, and the moon, and the neglected part of the earth's gravitational potential, corresponding to (1.03) minus (1.00). Since the expansion of (1.00) in zonal harmonics is

$$V'=-\mu r^{-1}\sum_{m=1}^{\infty}(r_e/r)^{2m}(-J_2)^m P_{2m}(\sin\theta), \tag{2.00}$$

this difference is

$$V-V'=\mu r^{-1}\left[\left(\frac{r_e}{r}\right)^3 J_3P_3(\sin\theta)+\left(\frac{r_e}{r}\right)^4(J_4+J_2^2)P_4(\sin\theta)+\left(\frac{r_e}{r}\right)^5 J_5P_5(\sin\theta)\right.$$
$$\left.+\left(\frac{r_e}{r}\right)^6(J_6-J_2^3)P_6(\sin\theta)+\ldots\right]+\text{tesseral harmonics}. \tag{2.01}$$

Of these forces the most important, for any satellite with a large ratio of mass to area, are the forces corresponding to $J_3$ and $J_4+J_2^2$ in (2.01) and drag, which as determined empirically may include effects of meteoritic impact. For a double satellite [Langer and Vinti 1963] only (2.01) and the lunar-solar perturbation remain.

The purpose of the present paper is to devise a method for correcting for the effects of any of the *zonal* harmonics in (2.01). The first example considered is the residual fourth harmonic, with coefficient $J_4+J_2^2$. This harmonic leads not only to short-periodic effects and secular effects, but also to long-periodic effects depending on a resonance denominator $1-5\cos^2 I$, giving rise to a critical inclination $I=63.4°$. The second example considered is the third harmonic, with coefficient $J_3$. This gives rise only to short-periodic effects and to long-periodic effects without singularities, so that it is qualitatively less interesting. Because of its greater magnitude, however, $J_3$ being about $(-2.4)10^{-6}$ and $|J_4+J_2^2|$ being probably somewhat less than $(0.5)10^{-6}$ [Kaula 1962; King-Hele, Cook, and Rees 1963], it leads to somewhat larger periodic effects.

## 3. The Dynamical Problem

Our problem is thus to find the motion of a satellite, taken to be of unit mass, when the Hamiltonian is

$$F = -T + \mu\rho(\rho^2 + c^2\eta^2)^{-1} + F_1, \tag{3.01}$$

where $T$ is its kinetic energy and where, for the residual fourth harmonic

$$F_1 = -\mu r_e^4 r^{-5}\sigma_4 P_4(\sin\theta) \tag{3.02}$$

$$\sigma_4 \equiv J_4 + J_2^2 \tag{3.03}$$

(We have here reversed the sign of the Hamiltonian, to agree with the usual practice with Delaunay variables.) In carrying out this solution we shall use the results and notation of the solution [Vinti 1961 a,b, 1962] for the reference orbit, for which $F' = 0$. If in (3.02) we then put

$$r = a(1 - e\cos E) = a(1 - e^2)(1 + e\cos v)^{-1} \tag{3.04}$$

$$\sin\theta = \sin I\sin(v + \beta_2), \tag{3.05}$$

the expression (3.02) for $F_1$ will be correct through order $J_2^2$. To this order of accuracy the anomalies $E$ and $v$ may be given by the quasi-elliptic expressions

$$E \approx M_s + E_0 \qquad (3.06) \qquad v \approx M_s + v_0. \tag{3.07}$$

Note that (3.04) corresponds to $r \approx \rho$ and (3.05) to other approximations of zeroth order in $J_2$, viz,

$$\psi \approx \psi_s + \psi_0 \approx M_s + \beta_2 + v_0 \approx v + \beta_2 \qquad (3.08) \qquad \eta \approx \sin\theta. \tag{3.09}$$

Such an order of accuracy will result in errors of order $J_2^3$ for those secular and short-periodic effects which are produced by the perturbing potential (3.02) and of order $J_2^2$ for the corresponding long-periodic effects. This perturbation (3.02) represents about 0.1 percent of the departure of the earth from sphericity. The solution for the other harmonics in (2.01) will have the same accuracy. However, since all of these higher harmonics represent only about 0.5 percent of the earth's departure from sphericity, their lower accuracy, as compared with that of the reference orbit which has already accounted for 99.5 percent of this departure, should not result in serious cumulative errors.

In doing the perturbation theory, the first canonical variables that come to mind are the Jacobi "constants," viz, the $\alpha$'s and $\beta$'s of the reference orbit. When the reference orbit is elliptic, however, their shortcomings are well known and they lead to the same troubles in the present problem, giving rise to Poisson terms, linear in the time, in the variations in $\alpha_1$ and $\alpha_2$.

The next set of canonical variables that one might try is the set generated from the $\alpha$'s and $\beta$'s by the generating function

$$S' = -\alpha_1 t + \mu(-2\alpha_1)^{-\frac{1}{2}}\beta_1' + \alpha_2\beta_2' + \alpha_3\beta_3'. \tag{3.10}$$

If we define $n_0$ by

$$\mu = n_0^2 a_0^3 \qquad a_0 \equiv -\tfrac{1}{2}\mu/\alpha_1, \tag{3.11}$$

the resulting canonical variables are

$$\begin{aligned} L &= (\mu a_0)^{1/2} \qquad l = n_0(t+\beta_1) \\ \alpha_2 &\qquad\qquad \beta_2 \\ \alpha_3 &\qquad\qquad \beta_3, \end{aligned} \tag{3.12}$$

canonical with respect to the Hamiltonian

$$F = \tfrac{1}{2}\mu^2/L^2 + F_1. \tag{3.13}$$

When the reference orbit is elliptic, this set is the same as the fast Delaunay set [Garfinkel 1960].

One may then attempt to apply the von Zeipel method in the way successfully used by Brouwer [1959] and Garfinkel [1959], first eliminating short-periodic terms and then proceeding to eliminate long-periodic terms. One finds, however, that the corresponding generating function $S_1^*$, which ought to be of the first order in the parameter $\sigma_4 \equiv J_4 + J_2^2$, must then satisfy

$$\partial S_1^*/\partial \beta_1' = \text{zeroth order in } \sigma_4. \tag{3.14}$$

One may alternatively eliminate short-periodic and long-periodic terms simultaneously, but one then obtains a Poisson term of the form $v' \sin 2\beta_2'$ in $\alpha_2 - \alpha_2'$. Since $v'$ has a secular part, such a result would appear absurd, since the "constant" $\alpha_2$, which ought to have only a small periodic variation, would then increase indefinitely with time.

These difficulties are examples of the failure of the von Zeipel method whenever the following conditions *both* hold: (1) the perturbing potential has a long-periodic part of the first order in the perturbation parameter, and (2) the canonical variables are such that the unperturbed Hamiltonian depends only on $L$.

To obtain a successful set of variables, we may proceed as follows. Let $q_i^0$ and $p_i^0$, $i = 1,2,3$, be the coordinates and momenta $\rho$, $\eta$, $\phi$, $p_\rho$, $p_\eta$, $p_\phi$ corresponding to the unperturbed problem (the reference orbit), with Hamiltonian $F = F_0$. Also let $j_i^0$, $w_i^0$, $i = 1,2,3$, be the corresponding action and angle variables.

Then

$$p_i^0 = \partial S(q_1^0, q_2^0, q_3^0, j_1^0, j_2^0, j_3^0)/\partial q_i^0 \tag{3.15}$$

$$w_i^0 = \partial S(q_1^0, q_2^0, q_3^0, j_1^0, j_2^0, j_3^0)/\partial j_i^0, \qquad (i = 1,2,3) \tag{3.16}$$

where $S$ is the Hamilton-Jacobi function of the unperturbed problem [Vinti 1959b], with the Jacobi $\alpha$'s replaced by the $j_i^0$. Here

$$j_i^0 \equiv \oint p_i^0 dq_i^0 = j_i^0(\alpha_1, \alpha_2, \alpha_3). \tag{3.17}$$

Now let $q_i$, $p_i$, $i = 1,2,3$, be the coordinates and momenta corresponding to the perturbed problem, with Hamiltonian $F = F_0 + F_1$. Introduce new variables $j_i$, $w_i$, $i = 1,2,3$, by means of the canonical transformation

$$p_i = \partial S(q_1, q_2, q_3, j_1, j_2, j_3)/\partial q_i \tag{3.18}$$

$$w_i = \partial S(q_1, q_2, q_3, j_1, j_2, j_3)/\partial j_i, \qquad (i = 1,2,3) \tag{3.19}$$

where $S$ is the same function of the $q_i$ and $j_i$ that the above Hamilton-Jacobi function is of the $q_i^0$ and $j_i^0$. Then the $w_i$ and $j_i$ are canonical variables, satisfying the equations

$$dj_i/dt = \partial F/\partial w_i \neq 0 \tag{3.20}$$

$$dw_i/dt = -\partial F/\partial j_i \neq \text{constant} \tag{3.21}$$

They are thus not action and angle variables, since the $j_i$ are not constant and the $w_i$ are not linear functions of $t$. Moreover

$$j_i \neq \oint p_i dq_i, \tag{3.22}$$

in contradistinction with (3.17).

It pays to go further, however, and introduce still another set of variables, a new Delaunay set $L, G, H, l, g, h$, by the transformations

$$\begin{aligned} 2\pi L &= j_1 + j_2 + j_3 \text{ sgn } \alpha_3 \\ 2\pi G &= j_2 + j_3 \text{ sgn } \alpha_3 \\ 2\pi H &= j_3 \end{aligned} \tag{3.23}$$

$$\begin{aligned} l &= 2\pi w_1 \\ g &= 2\pi(w_2 - w_1) \\ h &= 2\pi(w_3 - w_2 \text{ sgn } \alpha_3), \end{aligned} \tag{3.24}$$

where sgn $\alpha_3 = \pm 1$ respectively for a direct orbit or a retrograde orbit. To verify that they are canonical, note that

$$Ldl + Gdg + Hdh = j_1 dw_1 + j_2 dw_2 + j_3 dw_3. \tag{3.25}$$

They were used by Izsak [1962] in his application of the author's reference orbit to the problem of the critical inclination. From (3.23) we now have

$$j_1 = 2\pi(L - G) \qquad j_2 = 2\pi(G - H \text{ sgn } \alpha_3) \qquad j_3 = 2\pi H. \tag{3.26}$$

## 4. The New Delaunay Set

The functional relations among any of the quantities $\alpha_i$, $j_i$, $w_i$ for the perturbed problem are the same as those connecting $\alpha_i^0$, $j_i^0$, $w_i^0$ for the unperturbed problem. We may therefore usually drop superscript zeros and depend on the context for the meanings of the quantities.

From the author's paper [Vinti 1959 b] we now find

$$j_1 = 2\int_{\rho_1}^{\rho_2} \frac{\partial S}{\partial \rho} d\rho = 2\pi[\mu(-2\alpha_1)^{-1/2} - \alpha_2] + O(J_2) \tag{4.00}$$

$$j_2 = 4\int_0^{\eta_0} \frac{\partial S}{\partial \eta} d\eta = 2\pi(\alpha_2 - \alpha_3 \text{ sgn } \alpha_3) + O(J_2) \tag{4.01}$$

$$j_3 = 2\pi\alpha_3. \tag{4.02}$$

Next put

$$\alpha_{rs} \equiv \partial\alpha_r/\partial j_s \tag{4.03}$$

$$\nu_i \equiv \partial\alpha_1^0/\partial j_i^0 \tag{4.04}$$

$$j_{rs} \equiv \partial j_r/\partial\alpha_s. \tag{4.05}$$

Then, since

$$\sum_{i=1}^{3} \frac{\partial S}{\partial j_i}\frac{\partial j_i}{\partial \alpha_k} = \frac{\partial S}{\partial \alpha_k}, \qquad (k = 1,2,3) \tag{4.06}$$

we find

$$\frac{\partial S}{\partial \alpha_1} = t + \beta_1 = j_{11}w_1 + j_{21}w_2 \tag{4.07}$$

$$\frac{\partial S}{\partial \alpha_2} = \beta_2 = j_{12}w_1 + j_{22}w_2 \tag{4.08}$$

$$\frac{\partial S}{\partial \alpha_3} = \beta_3 = j_{13}w_1 + j_{23}w_2 + 2\pi w_3. \tag{4.09}$$

With the aid of (3.24) these equations become

$$2\pi(t+\beta_1)=j_{11}l+j_{21}(l+g) \tag{4.10}$$

$$2\pi\beta_2=j_{12}l+j_{22}(l+g) \tag{4.11}$$

$$2\pi\beta_3=j_{13}l+(j_{23}+2\pi\ \mathrm{sgn}\ \alpha_3)(l+g)+2\pi h. \tag{4.12}$$

The quantities $j_{rs}$ occurring here are given explicitly as functions of the $\alpha$'s in eqs (7.16) through (7.21) of an earlier paper [Vinti 1961a].

The constant orbital elements in the perturbed problem are then the constant parts $a''$, $e''$, $\eta_0''$ of $a,e,\eta_0$, along with the initial values $l_0''$, $g_0''$, $h_0''$ of the secular parts of $l$, $g$, $h$. The corresponding Hamiltonian $F$ is given by

$$F=F_0(L,G,H)+F_1 \qquad (4.13) \qquad F_0=-\alpha_1, \tag{4.14}$$

so that

$$\dot{L}=\partial F/\partial l \qquad\qquad \dot{l}=-\partial F/\partial L$$

$$\dot{G}=\partial F/\partial g \qquad (4.15) \qquad \dot{g}=-\partial F/\partial G \tag{4.16}$$

$$\dot{H}=\partial F/\partial h=0. \qquad\qquad \dot{h}=-\partial F/\partial H.$$

With the use of (3.23), (3.26), and (4.04), we find for the *unperturbed problem* that

$$\dot{l}=-\frac{\partial F_0}{\partial L}=\frac{\partial\alpha_1}{\partial L}=\sum_{i=1}^{3}\frac{\partial\alpha_1}{\partial j_i}\frac{\partial j_i}{\partial L}=2\pi\nu_1$$

$$\dot{g}=-\frac{\partial F_0}{\partial G}=\frac{\partial\alpha_1}{\partial G}=\sum_{i=1}^{3}\frac{\partial\alpha_1}{\partial j_i}\frac{\partial j_i}{\partial G}=2\pi(\nu_2-\nu_1) \tag{4.17}$$

$$\dot{h}=-\frac{\partial F_0}{\partial H}=\frac{\partial\alpha_1}{\partial H}=\sum_{i=1}^{3}\frac{\partial\alpha_1}{\partial j_i}\frac{\partial j_i}{\partial H}=2\pi(\nu_3-\nu_2\ \mathrm{sgn}\ \alpha_3)$$

and thus for the unperturbed problem that

$$\begin{aligned} l&=l_0+2\pi\nu_1 t\\ g&=g_0+2\pi(\nu_2-\nu_1)t\\ h&=h_0+2\pi(\nu_3-\nu_2\ \mathrm{sgn}\ \alpha_3)t, \end{aligned} \tag{4.18}$$

where we have dropped the double primes from $l_0$, $g_0$, $h_0$.

Before finding the effects of the perturbing potential, it is desirable to change the algorithm for the unperturbed problem, given in an earlier paper [Vinti 1961 a, pp. 197–200], so that the constant orbital elements become $a$, $e$, $I$, $l_0$, $g_0$, and $\beta_3$. ($\beta_3$ is better than $h_0$, as we shall see later.) To do so, insert (4.10), (4.11), and (4.18) into eqs (8.2) and (8.3) of that paper and carry out the same process that was carried out there. One finds

$$M_s=l_0+2\pi\nu_1 t \tag{4.19}$$

$$\psi_s=l_0+g_0+2\pi\nu_2 t. \tag{4.20}$$

With these new definitions of $M_s$ and $\psi_s$ the algorithm then becomes the same as in the earlier paper. Note, however, that the restriction on the angle of inclination $I$ has been removed [Vinti 1962].

## 5. Variations in the Spheroidal Coordinates

It is convenient to derive here the variations in $\rho$, $\eta$, and $\phi$ that will arise from the variations produced by the perturbing potential in the Delaunay variables. From the later solution of the

canonical equations (4.15) and (4.16) we shall find that the $m$th zonal harmonic produces variations in the Delaunay variables of the forms

$$\begin{aligned} &\delta L = \sigma_m L_m && \delta l = \sigma_m l_m + \sigma_m J_2^{-1} \tilde{l}_m \\ &\delta G = \sigma_m G_m + \sigma_m J_2^{-1} \tilde{G}_m \qquad (5.00) && \delta g = \sigma_m g_m + \sigma_m J_2^{-1} \tilde{g}_m \qquad (5.01) \\ &\delta H = 0. && \delta h = \sigma_m h_m + \sigma_m J_2^{-1} \tilde{h}_m. \end{aligned}$$

Here $\sigma_3 = J_3$ and $\sigma_4 = J_4 + J_2^2$. The terms $\sigma_m L_m$ and $\sigma_m G_m$ are short-periodic of order $J_2^2$; the products of $\sigma_m J_2^{-1}$ with $\tilde{G}_m$, $\tilde{l}_m$, $\tilde{g}_m$, and $\tilde{h}_m$ are long-periodic of order $J_2$. For $m = 3$ the products of $\sigma_3$ with $l_3$, $g_3$, and $h_3$ are short-periodic of order $J_2^2$. For $m = 4$

$$\begin{aligned} l_4 &= l_{41} + l_{42} \\ g_4 &= g_{41} + g_{42} \qquad (5.02) \\ h_4 &= h_{41} + h_{42}, \end{aligned}$$

where the products of $\sigma_4$ with $l_{41}$, $g_{41}$, and $h_{41}$ are short-periodic terms of order $J_2^2$ and where the products of $\sigma_4$ with $l_{42}$, $g_{42}$, and $h_{42}$ are secular terms of order $J_2^2$.

From $\delta L$ and $\delta G$ we can find the variations of the $j$'s, then of the $\alpha$'s, and finally of the elements $a$,$e$, and $\eta_0 \equiv \sin I$. From these and from $\delta l$, $\delta g$, and $\delta h$ we can then find $\delta E$, $\delta v$, $\delta \psi$, and $\delta \chi$ and finally the coordinate variations $\delta \rho$, $\delta \eta$, and $\delta \phi$. To obtain the $\delta j$'s, use (3.26) and (5.00). The results are

$$\begin{aligned} \delta j_1 &= 2\pi(\sigma_m L_m - \sigma_m G_m - \sigma_m \tilde{G}_m / J_2) \\ \delta j_2 &= 2\pi(\sigma_m G_m + \sigma_m \tilde{G}_m / J_2) \qquad (5.03) \\ \delta j_3 &= 0. \end{aligned}$$

To find the $\delta\alpha$'s use

$$\delta \alpha_r = \sum_{i=1}^{3} \frac{\partial \alpha_r}{\partial j_i} \delta j_i. \qquad (5.04)$$

Within the accuracy of the calculation, the coefficients $\partial \alpha_r / \partial j_i$ are needed only through order $J_2^0$. Thus by (5.03), (5.04), and (4.04)

$$\delta \alpha_1 = 2\pi \nu_1 \sigma_m L_m + 2\pi(\nu_2 - \nu_1)\sigma_m G_m + 2\pi(\nu_2 - \nu_1)\sigma_m \tilde{G}_m / J_2. \qquad (5.05)$$

Then, by (5.03), (5.04), and (4.03)

$$\delta \alpha_2 = 2\pi \alpha_{21} \sigma_m L_m + 2\pi(\alpha_{22} - \alpha_{21})\sigma_m G_m + 2\pi(\alpha_{22} - \alpha_{21})\sigma_m \tilde{G}_m / J_2. \qquad (5.06)$$

Also, by (4.02) and (5.03)

$$\delta \alpha_3 = 0. \qquad (5.07)$$

To find $\alpha_{21}$ and $\alpha_{22}$ note that

$$\begin{aligned} \sum_{i=1}^{3} \frac{\partial \alpha_2}{\partial j_i} \frac{\partial j_i}{\partial \alpha_2} &= \frac{\partial \alpha_2}{\partial \alpha_2} = 1 \\ \sum_{i=1}^{3} \frac{\partial \alpha_2}{\partial j_i} \frac{\partial j_i}{\partial \alpha_1} &= \frac{\partial \alpha_2}{\partial \alpha_1} = 0, \end{aligned} \qquad (5.08)$$

so that

$$\begin{aligned} \alpha_{21} j_{21} + \alpha_{22} j_{22} &= 1 \\ \alpha_{21} j_{11} + \alpha_{22} j_{12} &= 0, \end{aligned} \qquad (5.09)$$

with the solution

$$\alpha_{21} = -j_{21}/\Delta \qquad \alpha_{22} = j_{11}/\Delta, \tag{5.10}$$

where

$$\Delta \equiv j_{11}j_{22} - j_{12}j_{21}. \tag{5.11}$$

From an earlier paper [Vinti 1961 a, p 189], it then follows that

$$\alpha_{21} = \mathrm{O}(J_2) \qquad \alpha_{22} = (2\pi)^{-1} + \mathrm{O}(J_2) \tag{5.12}$$

and also that

$$\begin{aligned} 2\pi\nu_1 &= n + \mathrm{O}(J_2) \\ \nu_2 - \nu_1 &= \mathrm{O}(J_2), \end{aligned} \tag{5.13}$$

where

$$n \equiv \mu^{1/2} a^{-3/2}. \tag{5.14}$$

On carrying secular and short-periodic variations only through order $J_2^2$ and long-periodic perturbations only through order $J_2$, it then follows from (5.04) through (5.08) and (5.12) through (5.14) that

$$\delta\alpha_1 = n\sigma_m L_m \tag{5.15}$$

$$\delta\alpha_2 = \sigma_m G_m + \sigma_m \tilde{G}_m / J_2 \tag{5.16}$$

$$\delta\alpha_3 = 0. \tag{5.17}$$

In finding the variations of other quantities we may drop all terms in their defining equations of order $J_2^k$, where $k \geqq 1$. To show this, note that if $P = \mathrm{O}(J_2^0)$ and $\delta P = \sigma_m P_1 + \sigma_m J_2^{-1} \tilde{P}_1$ where $P_1$ and $\tilde{P}_1$ are both of order $J_2^0$, then

$$\delta(J_2^k P) = J_2^k(\sigma_m P_1 + \sigma_m J_2^{-1}\tilde{P}_1). \tag{5.18}$$

Thus if $k \geqq 1$, $J_2^k\, \sigma_m P_1$ is a secular plus short-periodic term of order $k+2 \geqq 3$ and $J_2^k\, \sigma_m J_2^{-1}\tilde{P}_1$ is a long-periodic term of order $k+1 \geqq 2$. This proves the statement.

To obtain the variations of $a$, $e$, and $\eta_0$ we may thus use

$$a \approx -\frac{\mu}{2\alpha_1} \qquad 1 - e^2 \approx -\frac{2\alpha_1\alpha_2^2}{\mu^2} \qquad 1 - \eta_0^2 \approx \frac{\alpha_3^2}{\alpha_2^2}, \tag{5.19}$$

the neglected terms being of order $J_2$. There follow

$$\begin{aligned} \delta a &= \frac{\mu}{2\alpha_1^2}\,\delta\alpha_1 = \frac{2a^2}{\mu}\,\delta\alpha_1 \\ e\delta e &= \frac{\alpha_2^2}{\mu^2}\,\delta\alpha_1 + \frac{2\alpha_1\alpha_2}{\mu^2}\,\delta\alpha_2 \\ \eta_0\delta\eta_0 &= \frac{\alpha_3^2}{\alpha_2^3}\,\delta\alpha_2. \end{aligned} \tag{5.20}$$

With use of (5.15), (5.16), (5.19), (5.20) and the relation

$\alpha_2^2 = \mu p + \mathrm{O}(J_2)$, we then find

$$\delta a = \frac{2\sigma_m L_m}{an} \tag{5.21}$$

$$\delta e = \frac{pn}{\mu e}\,\sigma_m L_m - (ae)^{-1}\left(\frac{p}{\mu}\right)^{1/2}(\sigma_m G_m + \sigma_m J_2^{-1}\tilde{G}_m) \tag{5.22}$$

$$\delta\eta_0 = \eta_0^{-1}(\mu p)^{-1/2}(1 - \eta_0^2)(\sigma_m G_m + \sigma_m J_2^{-1}\tilde{G}_m). \tag{5.23}$$

The element $a$ thus has only a short-periodic variation while the elements $e$ and $\eta_0 \equiv \sin I$ have both short-periodic and long-periodic but no secular variations.

To find $\delta E$, $\delta v$, and $\delta\psi$ we first insert (4.10) and (4.11) into eqs (8.2) and (8.3) of the earlier paper [Vinti 1961 a], rejecting all terms of order $J_2^k$, where $k \geqq 1$. We find

$$E - e \sin E = l + O(J_2) \tag{5.24}$$

$$\psi = v + g + O(J_2) \tag{5.25}$$

Equation (5.24) gives

$$\delta E = (1 - e \cos E)^{-1}(\delta l + \sin E \delta e). \tag{5.26}$$

To find $\delta v$, we use the anomaly connection

$$\tan \frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan \frac{E}{2}, \tag{5.27}$$

from which there follows

$$\begin{aligned} \delta v &= \left(\frac{1+e}{1-e}\right)^{1/2}\left(1 - \frac{2e}{1+e}\sin^2\frac{v}{2}\right)\delta E + \frac{\sin \iota}{1-e^2}\delta e \\ &= (1-e^2)^{1/2}(1 - e\cos E)^{-1}[\delta E + (1-e^2)^{-1}\sin E \delta e]. \end{aligned} \tag{5.28}$$

Then

$$\delta\psi = \delta v + \delta g. \tag{5.29}$$

The variations of the spheroidal coordinates

$$\rho = a(1 - e\cos E) \tag{5.30}$$

$$\eta = \eta_0 \sin\psi \tag{5.31}$$

are then

$$\delta\rho = (1 - e\cos E)\delta a - a\cos E \delta e + ae \sin E \delta E \tag{5.32}$$

$$\delta\eta = \sin\psi \delta\eta_0 + \eta_0 \cos\psi \delta\psi \tag{5.33}$$

To find the variation $\delta\phi$ of the right ascension, we note that by eq (8.49) of the earlier paper [Vinti 1961 a],

$$\phi = \beta_3 + \chi \operatorname{sgn}\alpha_3 + O(J_2), \tag{5.34}$$

where

$$\tan\chi = (1 - \eta_0^2)^{1/2}\tan\psi \tag{5.35}$$

$$\cos\chi = (1 - \eta_0^2\sin^2\psi)^{-1/2}\cos\psi. \tag{5.36}$$

Equation (5.36) was given in Vinti [1961 b]. Also by (4.12) and the results

$$j_{13} = O(J_2) \qquad j_{13} + 2\pi \operatorname{sgn}\alpha_3 = O(J_2) \tag{5.37}$$

from page 189 of Vinti [1961 a], we find

$$\beta_3 = h + \mathrm{O}(J_2), \tag{5.38}$$

so that

$$\delta\beta_3 = \delta h. \tag{5.39}$$

From (5.35) and (5.36) we find

$$\delta\chi = (1-\eta_0^2\sin^2\psi)^{-1}[(1-\eta_0^2)^{1/4}\,\delta\psi - \tfrac{1}{2}(1-\eta_0^2)^{-1/4}\,\eta_0\sin 2\psi\delta\eta_0]. \tag{5.40}$$

Then, from (5.34), (5.39), (5.40), and (5.23), it follows that

$$\delta\phi = \delta h + (1-\eta_0^2\sin^2\psi)^{-1}\cos I[\delta\psi - \tfrac{1}{2}(\mu p)^{-1/4}\sin 2\psi(\sigma_m G_m + \sigma_m J_2^{-1}\tilde{G}_m)], \tag{5.41}$$

where we have used $(1-\eta_0^2)^{1/4}\operatorname{sgn}\alpha_3 = \cos I$.

It is well to note here that $\beta_3$ is just as useful an orbital element as $h_0$.

## 6. Solution for the Perturbed Delaunay Variables

We now have to solve the canonical equations (4.13) through (4.16), the perturbing potential being

$$F_1 = -\mu r e^4 r^{-5}\sigma_4 P_4(\sin\theta) \tag{6.00}$$

in the case of the residual fourth harmonic. To obtain secular and short-periodic variations through order $J_2^2$ and long-periodic variations through order $J_2$, it will suffice to use elliptic approximations in (6.00), since $\sigma_4 \equiv J_4 + J_2^2 = \mathrm{O}(J_2^2)$. Thus in (6.00) we may put

$$\sin\theta = \eta = \eta_0\sin\psi = \eta_0\sin(v+g) \tag{6.01}$$

$$r = \rho = a(1-e\cos E) = a(1-e^2)(1+e\cos v)^{-1} \tag{6.02}$$

and we may use

$$l = E - e\sin E \tag{6.03}$$

$$dl = (r/a)^2(1-e^2)^{-1/2}\,dv \tag{6.04}$$

$$\psi = v + g. \tag{6.05}$$

Then

$$F_1 = \frac{-\mu r_e^4\sigma_4}{8a^5}\left[\left(3-15\eta_0^2+\frac{105}{8}\eta_0^4\right)\left(\frac{a}{r}\right)^5 + 5\eta_0^2\left(3-\frac{7}{2}\eta_0^2\right)\left(\frac{a}{r}\right)^5\cos 2\psi + \frac{35}{8}\eta_0^4\left(\frac{a}{r}\right)^5\cos 4\psi\right]. \tag{6.06}$$

The part of $F_1$ independent of $l$ is then

$$F_{1m} \equiv (2\pi)^{-1}\int_0^{2\pi} F_1\,dl = -\frac{\mu r_e^4\sigma_4}{16\pi a^5}(1-e^2)^{-1/2}\int_0^{2\pi}\left[\left(3-15\eta_0^2+\frac{105}{8}\eta_0^4\right)\left(\frac{a}{r}\right)^3 + 5\eta_0^2\left(3-\frac{7}{2}\eta_0^2\right)\left(\frac{a}{r}\right)^3\cos 2\psi + \frac{35}{8}\eta_0^4\left(\frac{a}{r}\right)^3\cos 2\psi\right]dv \tag{6.07}$$

From (6.02) and (6.05) it follows that

$$\int_0^{2\pi} \left(\frac{a}{r}\right)^3 dv = 2\pi(1-e^2)^{-3}\left(1+\frac{3}{2}e^2\right) \tag{6.08}$$

$$\int_0^{2\pi} \left(\frac{a}{r}\right)^3 \cos 2\psi \, dv = \frac{3}{2}\pi e^2(1-e^2)^{-3} \cos 2g \tag{6.09}$$

$$\int_0^{2\pi} \left(\frac{a}{r}\right)^3 \cos 4\psi \, dv = 0. \tag{6.10}$$

Equations (6.07) through (6.10) then result in

$$F_{1m} = F_{1c} + F_{1p}, \tag{6.11}$$

where the constant part is

$$F_{1c} = -\frac{\mu r_e^4 \sigma_4}{8a^5}(1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)\left(3-15\eta_0^2+\frac{105}{8}\eta_0^4\right) \tag{6.12}$$

and the long-periodic part is

$$F_{1p} = -\frac{15\mu r_e^4 \sigma_4}{32a^5}(1-e^2)^{-7/2}e^2\eta_0^2\left(3-\frac{7}{2}\eta_0^2\right)\cos 2g \tag{6.13}$$

The short-periodic part is then given by

$$F_{1l} = F_1 - F_{1c} - F_{1p}, \tag{6.14}$$

so that

$$\begin{aligned} F_{1l} = -\frac{\mu r_e^4 \sigma_4}{8a^5}\Bigg\{ & \left(3-15\eta_0^2+\frac{105}{8}\eta_0^4\right)\left[\left(\frac{a}{r}\right)^5 - (1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)\right] \\ & + 5\eta_0^2\left(3-\frac{7}{2}\eta_0^2\right)\left[\left(\frac{a}{r}\right)^5 \cos 2\psi - \frac{3}{4}e^2(1-e^2)^{-7/2}\cos 2g\right] + \frac{35}{8}\eta_0^4\left(\frac{a}{r}\right)^5 \cos 4\psi\Bigg\}. \end{aligned} \tag{6.15}$$

From (3.23) and (4.00) through (4.02) we now obtain

$$L = \mu(-2\alpha_1)^{-1/2} \qquad G = \alpha_2 \qquad H = \alpha_3, \tag{6.16}$$

with neglect of terms of order $J_2$. To the same accuracy

$$a = -\tfrac{1}{2}\mu\alpha_1^{-1} \qquad \alpha_2^2 = \mu a(1-e^2) \qquad \alpha_3^2\alpha_2^{-2} = 1-\eta_0^2. \tag{6.17}$$

Thus, with neglect of terms of order $J_2$,

$$a = L^2\mu^{-1} \qquad 1-e^2 = G^2L^{-2} \qquad \eta_0^2 = 1-H^2G^{-2}, \tag{6.18}$$

as expected. When we later take derivatives of a generating function with respect to $L$, $G$, and $H$, we shall have to use the expressions (6.18) to replace them in the results by the elements $a$, $e$, and $\eta_0$.

In solving the canonical equations (4.15) and (4.16) we first make a canonical transformation to new canonical variables $L'$, $G'$, $H'$, $l'$, $g'$, and $h'$, so that the new Hamiltonian $F^*$ shall be independent of $l'$ and $h'$. This first step will yield the short-periodic effects. To carry it out, introduce the generating function

$$S(L', G', H', l, g, h) = S_0 + S_1(L', G', H', l, g), \tag{6.19}$$

where

$$S_0 \equiv L'l + G'g + H'h \tag{6.20}$$

and where $S_1$ is to be of the first order in $\sigma_4$. Then follow the von Zeipel method, as applied by Brouwer (1959) to satellite orbits.

On splitting $F^*$ into parts $F_0^*$ and $F_1^*$, we then have

$$F_0(L, G, H) + F_1(L, G, H, l, g) = F_0^*(L', G', H') + F_1^*(L', G', H', g') \tag{6.21}$$

along with the following relations connecting $S_1$ and the primed and unprimed variables:

$$\begin{aligned} L &= \frac{\partial S}{\partial l} = L' + \frac{\partial S_1}{\partial l} \\ G &= \frac{\partial S}{\partial g} = G' + \frac{\partial S_1}{\partial g} \\ H &= \frac{\partial S}{\partial h} = H' \end{aligned} \tag{6.22}$$

$$\begin{aligned} l' &= \frac{\partial S}{\partial L'} = l + \frac{\partial S_1}{\partial L'} \\ g' &= \frac{\partial S}{\partial G'} = g + \frac{\partial S_1}{\partial G'} \\ h' &= \frac{\partial S}{\partial H'} = h + \frac{\partial S_1}{\partial H} \end{aligned} \tag{6.23}$$

Since the new Hamiltonian will not contain $l'$ or $h'$, we shall have $H' = H =$ constant and $L' =$ constant. Insertion of (6.22) and (6.23) into (6.21) then gives a partial differential equation for $S_1$:

$$F_0\left(L' + \frac{\partial S_1}{\partial l}, G' + \frac{\partial S_1}{\partial g}, H\right) + F_1\left(L' + \frac{\partial S_1}{\partial l}, G' + \frac{\partial S_1}{\partial g}, H, l + \frac{\partial S_1}{\partial L'}, g + \frac{\partial S_1}{\partial G'}\right)$$
$$= F_0^*(L', G', H) + F_1^*\left(L', G', H, g + \frac{\partial S_1}{\partial G'}\right). \tag{6.24}$$

Taylor expansion in the neighborhood of $L'$, $G'$, $H$, $l$, and $g$, with rejection of terms beyond the first order in $\sigma_4$, then gives

$$F_0(L', G', H) + \frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_0}{\partial G'}\frac{\partial S_1}{\partial g} + F_1(L', G', H, l, g) = F_0^*(L', G', H) + F_1^*(L', G', H, g). \tag{6.25}$$

The zeroth order terms lead to

$$F_0(L', G', H) = F_0^*(L', G', H) \tag{6.26}$$

and the first order terms to

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_0}{\partial G'}\frac{\partial S_1}{\partial G'} + F_{1c}(L', G', H) + F_{1p}(L', G', H, g) + F_{1l}(L', G', H, l, g) = F_1^*(L', G', H, g') \tag{6.27}$$

In writing down (6.27) we have used (6.14) to express $F_1$ as a sum of the terms $F_{1c}$, $F_{1p}$, and $F_{1l}$ and we have replaced $g$ by $g'$ in $F_1^*$, a permissible step involving an error of order $\sigma_4^2$. In (6.27) the terms independent of $l$ then yield

$$F_1^* = F_{1c}(L', G', H) + F_{1p}(L', G', H, g), \tag{6.28}$$

so that (6.26) and (6.28) together provide the new Hamiltonian $F^*$. The remaining terms, depending on $l$, then yield

$$\frac{\partial F_0}{\partial L'}\frac{\partial S_1}{\partial l} + \frac{\partial F_0}{\partial G'}\frac{\partial S_1}{\partial g} = -F_{1l}(L', G', H, l, g), \tag{6.29}$$

a partial differential equation for $S_1$. With use of (4.17) it becomes

$$2\pi\nu_1\frac{\partial S_1}{\partial l}+2\pi(\nu_2-\nu_1)\frac{\partial S_1}{\partial g}=F_{1l}(L',G',H,l,g) \tag{6.30}$$

Since $\nu_2-\nu_1$ is of order $J_2$ and since $S_1$ is to be of order $\sigma_4\equiv J_4+J_2^2$, it follows that $2\pi(\nu_2-\nu_1)\partial S_1/\partial g$ will be of order $J_2^3$ and is thus to be rejected. Thus

$$2\pi\nu_1 S_1=\int F_{1l}(L',G',H,l,g)dl+\Phi(g). \tag{6.31}$$

To evaluate the integral in (6.31), apply (6.15), (6.02), (6.04), and (6.18). Since $F_{1l}$ has a factor $\sigma_4=O(J_2^2)$, we can make a number of approximations at this point and still achieve our desired accuracy. These are: drop the primes from $L'$ and $G'$ in (6.31), place

$$2\pi\nu_1=n\equiv\mu^{1/2}a^{-3/2}, \tag{6.32}$$

express $a$, $e$, and $\eta_0$ by means of (6.18), drop the primes from $L'$ and $G'$ in calculating $\partial S_1/\partial L'$ and $\partial S_1/\partial G'$, and finally use (6.18) again to replace $L$, $G$, and $H$ in the final formulas by $a$, $e$, and $\eta_0$.

We obtain

$$\begin{aligned}S_1=\sigma_4Q_1&\left[\left(1+\frac{3}{2}e^2\right)(v-l)+\left(3e+\frac{3}{4}e^3\right)\sin v+\frac{3}{4}e^2\sin 2v+\frac{e^3}{12}\sin 3v\right]\\&+\sigma_4Q_2\left[\frac{3}{4}e^2(v-l)\cos 2g+\frac{e^3}{8}\sin(v-2g)+\left(\frac{3}{2}e+\frac{3}{8}e^3\right)\sin(v+2g)\right.\\&\left.+\left(\frac{1}{2}+\frac{3}{4}e^2\right)\sin(2v+2g)+\left(\frac{e}{2}+\frac{e^3}{8}\right)\sin(3v+2g)+\frac{3e^2}{16}\sin(4v+2g)+\frac{e^3}{40}\sin(5v+2g)\right]\\&+\sigma_4Q_3\left[\frac{e^3}{8}\sin(v+4g)+\frac{3}{8}e^2\sin(2v+4g)+\left(\frac{e}{2}+\frac{e^3}{8}\right)\sin(3v+4g)\right.\\&\left.+\left(\frac{1}{4}+\frac{3}{8}e^2\right)\sin(4v+4g)+\left(\frac{3e}{10}+\frac{3e^3}{40}\right)\sin(5v+4g)+\frac{e^2}{8}\sin(6v+4g)+\frac{e^3}{56}\sin(7v+4g)\right]+\Phi(g),\end{aligned} \tag{6.33}$$

where $\Phi(g)$ is a constant of integration. Here

$$Q_l=\frac{-\mu r_e^4}{8na^5}\left(\frac{L}{G}\right)^7 q_l=-\frac{1}{8}\left(\frac{r_e}{p}\right)^4(\mu p)^{1/2}q_l, \tag{6.34}$$

where

$$q_1\equiv 3-15\eta_0^2+\frac{105}{8}\eta_0^4\qquad q_2\equiv 5\eta_0^2\left(3-\frac{7}{2}\eta_0^2\right)\qquad q_3\equiv\frac{35}{8}\eta_0^4. \tag{6.35}$$

With use of (6.18) and (6.33) we find

$$\frac{\mu L^7}{na^5}=\mu^4+O(J_2), \tag{6.36}$$

and

$$Q_1=\frac{-\mu^4r_e^4}{64G^7}\left(9-90\frac{H^2}{G^2}+105\frac{H^4}{G^4}\right) \tag{6.37}$$

$$Q_2=\frac{5\mu^4r_e^4}{16G^7}\left(1-\frac{H^2}{G^2}\right)\left(1-7\frac{H^2}{G^2}\right) \tag{6.38}$$

$$Q_3=\frac{-35\mu^4r_e^4}{64G^7}\left(1-\frac{H^2}{G^2}\right)^2, \tag{6.39}$$

through terms of order $J_2^2$.

To make $S_1$ purely short-periodic, one would have to choose $\Phi(g)$ in such a way that

$2\pi\bar{S}_1 \equiv \int_0^{2\pi} S_1 dl$ would vanish. It turns out that $\Phi(g)$ would then not vanish, but would have to be a long-periodic term, just cancelling a long-periodic term of order $J_2^2$ arising from the rest of the expression for $S_1$. The later calculation of long-periodic effects, however, will be accurate only through order $J_2$. If we arbitrarily drop $\Phi(g)$, whose calculation would be extremely laborious, the net effect will be only to leave in the short-periodic terms of $G$, $l$, $g$, and $h$ some long-periodic impurities of order $J_2^2$, not affecting the accuracy of the calculation.

The errors of the short-periodic terms will be of order $J_2^2$, for two reasons. First, the short-periodic terms of the reference orbit were calculated only through terms of order $J_2^2$; second, the present calculation makes use of elliptic approximations, so that a variation of the form $\sigma_4 f$ has an error of order $J_2$ in $f$ and thus of order $J_2^2$ in $\sigma_4 f$. For this second reason the secular corrections produced by the perturbing potential will also have errors of order $J_2^2$, even though the first omitted term in the von Zeipel equation (6.25) is of order $\sigma_4^2$ or $J_2^4$.

## 7. Short-Periodic Terms

It is now straightforward but tedious to calculate the short-periodic terms. From (6.22), (6.31), (6.32), and (6.15), we find, through terms of order $J_2^2$

$$\frac{\partial S_1}{\partial l} = L - L' = \sigma_4 L_4, \tag{7.00}$$

where

$$L_4 = -\frac{r_e^4}{8a^5 n}\left\{\left(3 - 15\eta_0^2 + \frac{105}{8}\eta_0^4\right)\left[\left(\frac{a}{r}\right)^5 - (1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)\right]\right.$$

$$\left. + 5\eta_0^2\left(3 - \frac{7}{2}\eta_0^2\right)\left[\left(\frac{a}{r}\right)^5 \cos(2v+2g) - \frac{3e^2}{4}(1-e^2)^{-7/2}\cos 2g\right] + \frac{35}{8}\eta_0^4\left(\frac{a}{r}\right)^5 \cos(4v+4g)\right\}. \tag{7.01}$$

From (6.22) and (6.33), we find

$$\frac{\partial S_1}{\partial g} = G - G' = \sigma_4 G_4, \tag{7.02}$$

where

$$G_4 = Q_2\left[-\frac{3}{2}e^2(v-l)\sin 2g - \frac{e^3}{4}\cos(v-2g) + \left(3e + \frac{3e^3}{4}\right)\cos(v+2g)\right.$$

$$\left. + \left(1+\frac{3}{2}e^2\right)\cos(2v+2g) + \left(e+\frac{e^3}{4}\right)\cos(3v+2g) + \frac{3e^2}{8}\cos(4v+2g) + \frac{e^3}{20}\cos(5v+2g)\right]$$

$$+ Q_3\left[\frac{1}{2}e^3\cos(v+4g) + \frac{3}{2}e^2\cos(2v+4g) + \left(2e+\frac{1}{2}e^3\right)\cos(3v+4g) + \left(1+\frac{3}{2}e^2\right)\cos(4v+4g)\right.$$

$$\left. + \left(\frac{6e}{5} + \frac{3e^3}{10}\right)\cos(5v+4g) + \frac{1}{2}e^2\cos(6v+4g) + \frac{e^3}{14}\cos(7v+4g)\right]. \tag{7.03}$$

Since the dependence of $S_1$ on $H$ is only through the $Q$'s, the calculation of

$$h - h' = -\frac{\partial S_1}{\partial H} = \sigma_4 h_{41} \tag{7.04}$$

is simple. First calculate the derivatives $\partial Q_i/\partial H$. From (6.37) through (6.39) and the sufficiently

accurate relations $H/G=\cos I$, $H^2/G^2=1-\eta_0^2$ and $G^2=\mu p$, we find

$$\frac{\partial Q_1}{\partial H}=\frac{15}{16}\left(\frac{r_e}{p}\right)^4(7\eta_0^2-4)\cos I$$

$$\frac{\partial Q_2}{\partial H}=\frac{5}{4}\left(\frac{r_e}{p}\right)^4(3-7\eta_0^2)\cos I \tag{7.05}$$

$$\frac{\partial Q_3}{\partial H}=\frac{35}{16}\left(\frac{r_e}{p}\right)^4\eta_0^2\cos I.$$

From (7.04), (6.33), and (7.05) we then obtain

$$\begin{aligned}h_{41}=&-\frac{5}{16}\left(\frac{r_e}{p}\right)^4\cos I\Big\{3(7\eta_0^2-4)\Big[\Big(1+\frac{3}{2}e^2\Big)(v-l)+\Big(3e+\frac{3e^3}{4}\Big)\sin v+\frac{3e^2}{4}\sin 2v+\frac{e^3}{12}\sin 3v\Big]\\&+4(3-7\eta_0^2)\Big[\frac{3}{4}e^2(v-l)\cos 2g+\frac{e^3}{8}\sin(v-2g)+\Big(\frac{3e}{2}+\frac{3e^3}{8}\Big)\sin(v+2g)\\&+\Big(\frac{1}{2}+\frac{3e^2}{4}\Big)\sin(2v+2g)+\Big(\frac{e}{2}+\frac{e^3}{8}\Big)\sin(3v+2g)+\frac{3e^2}{16}\sin(4v+2g)+\frac{e^3}{40}\sin(5v+2g)\Big]\\&+7\eta_0^2\Big[\frac{e^3}{8}\sin(v+4g)+\frac{3e^2}{8}\sin(2v+4g)+\Big(\frac{e}{2}+\frac{e^3}{8}\Big)\sin(3v+4g)+\Big(\frac{1}{4}+\frac{3e^2}{8}\Big)\sin(4v+4g)\\&+\Big(\frac{3e}{10}+\frac{3e^3}{40}\Big)\sin(5v+4g)+\frac{e^2}{8}\sin(6v+4g)+\frac{e^3}{56}\sin(7v+4g)\Big]\Big\}.\end{aligned} \tag{7.06}$$

To find $l-l'$ and $g-g'$, we use (6.23), (6.33), and (6.38) through (6.40). We may write (6.33) as

$$S_1=\sigma_4Q_1(G,H)f_1(e,v)+\sigma_4Q_2(G,H)f_2(e,v)+\sigma_4Q_3(G,H)f_3(e,v), \tag{7.07}$$

where

$$f_1(e,v)\equiv\Big(1+\frac{3}{2}e^2\Big)(v-l)+\Big(3e+\frac{3e^3}{4}\Big)\sin v+\frac{3}{4}e^2\sin 2v+\frac{e^3}{12}\sin 3v \tag{7.08}$$

$$\begin{aligned}f_2(e,v)\equiv&\frac{3}{4}e^2(v-l)\cos 2g+\frac{e^3}{8}\sin(v-2g)+\Big(\frac{3e}{2}+\frac{3e^3}{8}\Big)\sin(v+2g)\\&+\Big(\frac{1}{2}+\frac{3}{4}e^2\Big)\sin(2v+2g)+\Big(\frac{e}{2}+\frac{e^3}{8}\Big)\sin(3v+2g)+\frac{3e^2}{16}\sin(4v+2g)+\frac{e^3}{40}\sin(5v+2g)\end{aligned} \tag{7.09}$$

$$\begin{aligned}f_3(e,v)\equiv&\frac{e^3}{8}\sin(v+4g)+\frac{3e^2}{8}\sin(2v+2g)+\Big(\frac{e}{2}+\frac{e^3}{8}\Big)\sin(3v+4g)\\&+\Big(\frac{1}{4}+\frac{3e^2}{8}\Big)\sin(4v+4g)+\Big(\frac{3e}{10}+\frac{3e^3}{40}\Big)\sin(5v+4g)+\frac{e^2}{8}\sin(6v+4g)+\frac{e^3}{56}\sin(7v+4g).\end{aligned} \tag{7.10}$$

Then

$$\frac{\partial S_1}{\partial L}=\sigma_4\sum_{i=1}^{3}Q_i\Big(\frac{\partial f_i}{\partial e}+\frac{\partial f_i}{\partial v}\frac{\partial v}{\partial e}\Big)\frac{\partial e}{\partial L} \tag{7.11}$$

$$\begin{aligned}\frac{\partial S_1}{\partial G}&=\sigma_4\sum_{i=1}^{3}\frac{\partial Q_i}{\partial G}f_i+\sigma_4\sum_{i=1}^{3}Q_i\Big(\frac{\partial f_i}{\partial e}+\frac{\partial f_i}{\partial v}\frac{\partial v}{\partial e}\Big)\frac{\partial e}{\partial G}\\&=\sigma_4\sum_{i=1}^{3}\frac{\partial Q_i}{\partial G}f_i+\frac{\partial S_1}{\partial L}\frac{\partial e}{\partial G}\Big(\frac{\partial e}{\partial L}\Big)^{-1}\end{aligned} \tag{7.12}$$

From (6.37) through (6.39), with the same approximations used in obtaining (7.05), we find

$$\frac{\partial Q_1}{\partial G}=\frac{3}{64}\left(\frac{r_e}{p}\right)^4(136-500\eta_0^2+385\eta_0^4)$$

$$\frac{\partial Q_2}{\partial G}=-\frac{5}{16}\left(\frac{r_e}{p}\right)^4(12-82\eta_0^2+77\eta_0^4) \tag{7.13}$$

$$\frac{\partial Q_3}{\partial G}=\frac{35}{64}\left(\frac{r_e}{p}\right)^4\eta_0^2(11\eta_0^2-4).$$

Next we need the elliptic approximations

$$\frac{\partial e}{\partial L}=(\mu a)^{-1/2}e^{-1}(1-e^2) \tag{7.14}$$

$$\frac{\partial e}{\partial G}=-e^{-1}\left(\frac{1-e^2}{\mu a}\right)^{1/2} \tag{7.15}$$

$$\frac{\partial v}{\partial e}=\left(\frac{1}{1-e^2}+\frac{a}{r}\right)\sin v=(1-e^2)^{-1}\left(2\sin v+\frac{1}{2}e\sin 2v\right) \tag{7.16}$$

To compute the $\partial S_1/\partial L$ and $\partial S_1/\partial G$ we need finally the six derivatives $\partial f_i/\partial e$ and $\partial f_i/\partial v$, $i=1, 2, 3$. From (7.08) through (7.10) we find

$$\frac{\partial f_1}{\partial e}=3e(v-l)+\left(3+\frac{9e^2}{4}\right)\sin v+\frac{3}{2}e\sin 2v+\frac{e^3}{4}\sin 3v \tag{7.17}$$

$$\frac{\partial f_1}{\partial v}=1+\frac{3}{2}e^2+\left(3e+\frac{3e^3}{4}\right)\cos v+\frac{3}{2}e^2\cos 2v+\frac{e^3}{4}\cos 3v \tag{7.18}$$

$$\begin{aligned}\frac{\partial f_2}{\partial e}=&\frac{3}{2}e(v-l)\cos 2g+\frac{3e^2}{8}\sin(v-2g)+\left(\frac{3}{2}+\frac{9e^2}{8}\right)\sin(v+2g)\\&+\frac{3}{2}e\sin(2v+2g)+\left(\frac{1}{2}+\frac{3e^2}{8}\right)\sin(3v+2g)+\frac{3e}{8}\sin(4v+2g)+\frac{3e^2}{40}\sin(5v+2g)\end{aligned} \tag{7.19}$$

$$\begin{aligned}\frac{\partial f_2}{\partial v}=&\frac{3}{4}e^2\cos 2g+\frac{e^3}{8}\cos(v-2g)+\left(\frac{3}{2}e+\frac{3e^3}{8}\right)\cos(v+2g)+\left(1+\frac{3}{2}e^2\right)\cos(2v+2g)\\&+\left(\frac{3}{2}e+\frac{3e^3}{8}\right)\cos(3v+2g)+\frac{3e^2}{4}\cos(4v+2g)+\frac{e^3}{8}\cos(5v+2g)\end{aligned} \tag{7.20}$$

$$\begin{aligned}\frac{\partial f_3}{\partial e}=&\frac{3e^2}{8}\sin(v+4g)+\frac{3e}{4}\sin(2v+4g)+\left(\frac{1}{2}+\frac{3e^2}{8}\right)\sin(3v+4g)+\frac{3e}{4}\sin(4v+4g)\\&+\left(\frac{3}{10}+\frac{9e^2}{40}\right)\sin(5v+4g)+\frac{e}{4}\sin(6v+4g)+\frac{3e^2}{56}\sin(7v+4g)\end{aligned} \tag{7.21}$$

$$\begin{aligned}\frac{\partial f_3}{\partial v}=&\frac{e^3}{8}\cos(v+4g)+\frac{3e^2}{4}\cos(2v+4g)+\left(\frac{3e}{2}+\frac{3e^3}{8}\right)\cos(3v+4g)+\left(1+\frac{3e^2}{2}\right)\cos(4v+4g)\\&+\left(\frac{3e}{2}+\frac{3e^3}{8}\right)\cos(5v+4g)+\frac{3e^2}{4}\cos(6v+4g)+\frac{e^3}{8}\cos(7v+4g).\end{aligned} \tag{7.22}$$

There then follow

$$\begin{aligned}(1-e^2)\left(\frac{\partial f_1}{\partial e}+\frac{\partial f_1}{\partial v}\frac{\partial v}{\partial e}\right)=&3e(1-e^2)(v-l)+\left(5+\frac{3e^2}{2}-\frac{17e^4}{8}\right)\sin v+\left(5e-\frac{e^3}{4}\right)\sin 2v\\&+\left(\frac{5e^2}{2}-\frac{e^4}{16}\right)\sin 3v+\frac{5e^3}{8}\sin 4v+\frac{e^4}{16}\sin 5v\end{aligned} \tag{7.23}$$

$$\begin{aligned}(1-e^2)\left(\frac{\partial f_2}{\partial e}+\frac{\partial f_2}{\partial v}\frac{\partial v}{\partial e}\right)=&-\frac{e}{8}(14+5e^2)\sin 2g+\frac{3e}{2}(1-e^2)(v-l)\cos 2g\\&+\left(\frac{3e^2}{2}-\frac{9e^4}{32}\right)\sin(v-2g)+\frac{5e^3}{16}\sin(2v-2g)+\frac{e^4}{32}\sin(3v-2g)\end{aligned}$$

$$+\left(\frac{1}{2}-\frac{3}{2}e^2-\frac{19e^4}{16}\right)\sin(v+2g)+\left(\frac{3}{2}e-\frac{3}{2}e^3\right)\sin(2v+2g)+\left(\frac{3}{2}+e^2-\frac{5e^4}{16}\right)\sin(3v+2g)$$
$$+\left(\frac{17e}{8}+\frac{e^3}{4}\right)\sin(4v+2g)+\left(\frac{6e^2}{5}+\frac{3e^4}{160}\right)\sin(5v+2g)$$
$$+\frac{5e^3}{16}\sin(6v+2g)+\frac{e^4}{32}\sin(7v+2g) \qquad (7.24)$$

$$(1-e^2)\left(\frac{\partial f_3}{\partial e}+\frac{\partial f_3}{\partial v}\frac{\partial v}{\partial e}\right)=\frac{e^4}{32}\sin(v-4g)-\frac{5e^3}{16}\sin 4g-\left(\frac{3e^2}{4}+\frac{15e^4}{32}\right)\sin(v+4g)$$
$$-\left(e+\frac{11e^3}{8}\right)\sin(2v+4g)-\left(\frac{1}{2}+\frac{5}{4}e^2+\frac{7e^4}{16}\right)\sin(3v+4g)+\left(\frac{3e}{4}-\frac{3e^2}{4}\right)\sin(4v+4g)$$
$$+\left(\frac{13}{10}+\frac{21e^2}{20}-\frac{13e^4}{80}\right)\sin(5v+4g)+\left(2e+\frac{3e^3}{8}\right)\sin(6v+4g)$$
$$+\left(\frac{33e^2}{28}+\frac{9e^4}{224}\right)\sin(7v+4g)+\frac{5e^3}{16}\sin(8v+4g)+\frac{e^4}{32}\sin(9v+4g). \qquad (7.25)$$

From (6.34), (7.14), and (7.16) it follows that

$$Q_i\frac{\partial e}{\partial L}=-\frac{1}{8}\left(\frac{r_e}{p}\right)^4\left(\frac{1-e^2}{e}\right)^{3/2}q_i \qquad (i=1,2,3) \qquad (7.26)$$

and then from (6.23), (7.11), and (7.23) through (7.26) that

$$l-l'=\sigma_4 l_{41}, \qquad (7.27)$$

where

$$l_{41}=\frac{(1-e^2)^{1/2}}{2048}\left(\frac{r_e}{p}\right)^4\Big\{6(8-40\eta_0^2+35\eta_0^4)\Big[48(1-e^2)(v-l)+\frac{2}{e}(40+12e^2-17e^4)\sin v$$
$$+4(20-e^2)\sin 2v+e(40-e^2)\sin 3v+10e^2\sin 4v+e^3\sin 5v\Big]$$
$$-4\eta_0^2(7\eta_0^2-6)\Big[240(1-e^2)(v-l)\cos 2g+5e^3\sin(3v-2g)+50e^2\sin(2v-2g)$$
$$+15e(16-3e^2)\sin(v-2g)-20(14+5e^2)\sin 2g+\frac{10}{e}(8-24e^2-19e^4)\sin(v+2g)$$
$$+240(1-e^2)\sin(2v+2g)+\frac{10}{e}(24+16e^2-5e^4)\sin(3v+2g)+20(17+2e^2)\sin(4v+2g)$$
$$+3e(64+e^2)\sin(5v+2g)+50e^2\sin(6v+2g)+5e^3\sin(7v+2g)\Big]$$
$$+\eta_0^4\Big[35e^3\sin(v-4g)-350e^2\sin 4g-105e(8+5e^2)\sin(v+4g)$$
$$-140(8+11e^2)\sin(2v+4g)-\frac{70}{e}(8+20e^2+7e^4)\sin(3v+4g)+840(1-e^2)\sin(4v+4g)$$
$$+\frac{14}{e}(104+84e^2-13e^4)\sin(5v+4g)+140(16+3e^2)\sin(6v+4g)$$
$$+15e(88+3e^2)\sin(7v+4g)+350e^2\sin(8v+4g)+35e^3\sin(9v+4g)\Big]\Big\}. \qquad (7.28)$$

From (6.23), (7.12), (7.14), and (7.15), we find

$$g-g'=-(1-e^2)^{-1/2}(l-l')-\sigma_4\sum_{i=1}^{3}\frac{\partial Q_i}{\partial G_i}f_i. \qquad (7.29)$$

Then, from (7.13) and (7.08) through (7.10), we find

$$g - g' = \sigma_4 g_{41}, \tag{7.30}$$

where

$$\begin{aligned} g_{41} = &-(1-e^2)^{-\frac{1}{2}} l_{41} - \frac{1}{512}\left(\frac{r_e}{p}\right)^4 \Big\{ 2(136 - 500\eta_0^2 + 385\eta_0^4)[6(2+3e^2)(v-l) \\ &+ 9e(4+e^2)\sin v + 9e^2 \sin 2v + e^3 \sin 3v] - 2(12 - 82\eta_0^2 + 11\eta_0^4)[60e^2(v-l)\cos 2g \\ &+ 10e^3 \sin(v-2g) + 30e(4+e^2)\sin(v+2g) + 20(2+3e^2)\sin(2v+2g) \\ &+ 10e(4+e^2)\sin(3v+2g) + 15e^2 \sin(4v+2g) + 2e^3 \sin(5v+2g)] \\ &+ \eta_0^2(11\eta_0^2 - 4)[35e^3 \sin(v+4g) + 105e^2 \sin(2v+4g) + 35e(4+e^2)\sin(3v+4g) \\ &+ 35(2+3e^2)\sin(4v+4g) + 21e(4+3e^2)\sin(5v+4g) \\ &+ 35e^2 \sin(6v+4g) + 5e^3 \sin(7v+4g)] \Big\}. \end{aligned} \tag{7.31}$$

This concludes the solution for the short-periodic terms arising from the residual fourth harmonic, with coefficient $\sigma_4 \equiv J_4 + J_2^2$.

## 8. Long-Periodic Terms

By (6.21) and (6.28) the Hamiltonian $F^*$ is

$$F^* = F_0(L', G', H) + F_{1c}(L', G', H) + F_{1p}(L', G', H, g'), \tag{8.00}$$

short-periodic terms having been eliminated. We now try to find new canonical variables $L''$, $G''$, $H''$, $l''$, $g''$, and $h''$, corresponding to a new Hamiltonian $F_0^{**}(L'', G'', H'') + F_1^{**}(L'', G'', H'')$, so that

$$F_0(L', G', H) + F_{1c}(L', G', H) + F_{1p}(L', G', H, g') = F_0^{**}(L'', G'', H'') + F_1^{**}(L'', G'', H''). \tag{8.01}$$

If we can do so, then $L''$, $G''$, and $H''$ will be constants of the motion and

$$\dot{l}'' = -\frac{\partial F^{**}}{\partial L''} \qquad \dot{g}'' = -\frac{\partial F^{**}}{\partial G''} \qquad \dot{h}'' = -\frac{\partial F^{**}}{\partial H''}. \tag{8.02}$$

To find the necessary canonical transformation, we introduce the generating function

$$S^* = L''l' + G''g' + H''h' + S_1^*(L'', G'', H'', g') \tag{8.03}$$

where $S_1^*$ is to be of order $\sigma_4$. Then

$$\begin{aligned} L' &= \frac{\partial S^*}{\partial l'} = L'' & l'' &= \frac{\partial S^*}{\partial L''} = l' + \frac{\partial S_1^*}{\partial L''} \\ G' &= \frac{\partial S^*}{\partial g'} = G'' + \frac{\partial S_1^*}{\partial g'} \qquad (8.04) & g'' &= \frac{\partial S^*}{\partial G''} = g' + \frac{\partial S_1^*}{\partial G''} \qquad (8.05) \\ H &= H' = \frac{\partial S^*}{\partial h'} = H'' & h'' &= \frac{\partial S^*}{\partial H''} = h' + \frac{\partial S_1^*}{\partial H} \end{aligned}$$

Insertion of (8.04) and (8.05) into (8.01) leads to

$$F_0\left(L', G''+\frac{\partial S_1^*}{\partial g'}, H\right)+F_{1c}\left(L', G''+\frac{\partial S_1^*}{\partial g'}, H\right)+F_{1p}\left(L', G''+\frac{\partial S_1^*}{\partial g'}, H, g'+\frac{\partial S_1^*}{\partial G''}\right)$$
$$=F_0^{**}+F_1^{**}, \qquad (8.06)$$

whose Taylor expansion, with neglect of terms of order $\sigma_4^2$ or higher, is

$$F_0(L', G'', H)+\frac{\partial F_0}{\partial G''}\frac{\partial S_1^*}{\partial g'}+F_{1c}(L', G'', H)+F_{1p}(L', G'', H, g')=F_0^{**}+F_1^{**}. \qquad (8.07)$$

Splitting (8.07) into zeroth order and first order terms yields, respectively,

$$F_0^{**}=F_0(L', G'', H) \qquad (8.08)$$

$$\frac{\partial F_0}{\partial G''}\frac{\partial S_1^*}{\partial g'}+F_{1c}+F_{1p}=F_1^{**}. \qquad (8.09)$$

Resolution of (8.09) into constant and long-periodic terms then shows that

$$F_1^{**}=F_{1c}(L', G'', H) \qquad (8.10)$$

$$\frac{\partial F_0}{\partial G''}\frac{\partial S_1^*}{\partial g'}=-F_{1p}(L', G'', H, g'). \qquad (8.11)$$

With use of (4.17) and (6.13) and of double primes to denote quantities corresponding to $G''$, (8.11) becomes

$$2\pi(\nu_1''-\nu_2'')\frac{\partial S_1^*}{\partial g'}=\frac{15\mu r_e^4\sigma_4}{32a''^5}e''^2(1-e''^2)^{-7/2}\eta_0''^2\left(3-\frac{7}{2}\eta_0''^2\right)\cos 2g'. \qquad (8.12)$$

By eqs (7.34) and (7.37) of an earlier paper [Vinti 1961 a], we have

$$2\pi(\nu_1''-\nu_2'')=-\frac{3r_e^2J_2n}{4p^2}(5\cos^2 I-1)+O(J_2^2)$$
$$=\frac{3}{4}\frac{\mu^4r_e^2J_2}{L'^3G''^4}\left(1-\frac{5H^2}{G''^2}\right)+O(J_2^2). \qquad (8.13)$$

On inserting (8.13) into (8.12) and replacing elements by Delaunay variables, we find

$$\frac{\partial S_1^*}{\partial g'}=-\frac{5\mu^2r_e^2\sigma_4}{16J_2G''^3}\left(1-\frac{G''^2}{L'^2}\right)\left(1-\frac{H^2}{G''^2}\right)\left(1-7\frac{H^2}{G''^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}\cos 2g'. \qquad (8.14)$$

Integration then yields

$$S_1^*=-\frac{5\mu^2r_e^2\sigma_4}{32J_2G''^3}\left(1-\frac{G''^2}{L'^2}\right)\left(1-\frac{H^2}{G''^2}\right)\left(1-7\frac{H^2}{G''^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}\sin 2g'. \qquad (8.15)$$

Then

$$\frac{\partial S_1^*}{\partial L'}=-\frac{5\mu^2r_e^2\sigma_4}{16J_2G''L'^3}\left(1-\frac{H^2}{G''^2}\right)\left(1-7\frac{H^2}{G''^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}\sin 2g' \qquad (8.16)$$

$$\frac{\partial S_1^*}{\partial G''}=\frac{5\mu^2r_e^2\sigma_4}{32J_2G''^4}\left[3\left(1-5\frac{H^2}{G''^2}\right)-\frac{G''^2}{L'^2}\left(1-9\frac{H^2}{G''^2}\right)-\frac{8H^4}{G''^4}\left(7-5\frac{G''^2}{L'^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-1}\right.$$
$$\left.-80\frac{H^6}{G''^6}\left(1-\frac{G''^2}{L'^2}\right)\left(1-5\frac{H^2}{G''^2}\right)^{-2}\right]\sin 2g' \qquad (8.17)$$

$$\frac{\partial S_1^*}{\partial H}=\frac{5\mu^2 r_e^2\sigma_4}{16J_2}\frac{H}{G''^5}\left(1-\frac{G''^2}{L'^2}\right)\left[3+16\frac{H^2}{G''^2}\left(1-5\frac{H^2}{G''^2}\right)^{-1}+40\frac{H^4}{G''^4}\left(1-5\frac{H^2}{G''^2}\right)^{-2}\right]\sin 2g'. \tag{8.18}$$

We now express the long-periodic terms in the notation of (5.00) and (5.01).

$$G'-G''=\frac{\sigma_4}{J_2}\tilde{G}_4 \qquad g'-g''=\frac{\sigma_4}{J_2}\tilde{g}_4$$

$$l'-l''=\frac{\sigma_4}{J_2}\tilde{l}_4 \qquad h'-h''=\frac{\sigma_4}{J_2}\tilde{h}_4. \tag{8.19}$$

If we then use (8.04), (8.05), (8.14), and (8.16) through (8.19) and place $G''^2=\mu p$, $L'^2=\mu a$, $H/G''=\cos I$, $G''^2/L'^2=1-e^2$, $g'=$ unperturbed $g$ as given by (4.18), we find to the required accuracy

$$\tilde{G}_4=-\frac{5r_e^2 n}{16}e^2(1-e^2)^{-3/2}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\cos 2g \tag{8.20}$$

$$\tilde{l}_4=\frac{5}{16}\left(\frac{r_e}{a}\right)^2(1-e^2)^{-1/2}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\sin 2g \tag{8.21}$$

$$\tilde{g}_4=-\frac{5}{32}\left(\frac{r_e}{p}\right)^2[2+e^2-3(2+3e^2)\cos^2 I-8(2+5e^2)\cos^4 I(1-5\cos^2 I)^{-1}$$

$$-80e^2\cos^6 I(1-5\cos^2 I)^{-2}]\sin 2g \tag{8.22}$$

$$\tilde{h}_4=-\frac{5}{16}\left(\frac{r_e}{p}\right)^2 e^2\cos I[3+16\cos^2 I(1-5\cos^2 I)^{-1}+40\cos^4 I(1-5\cos^2 I)^{-2}]\sin 2g. \tag{8.23}$$

## 9. Secular Effects

We now have to use (8.02), (8.08), (8.10), and (4.17) to obtain $l''$, $g''$, and $h''$ as linear functions of the time. From (4.17) and (8.08) we obtain

$$\frac{\partial F_0^{**}}{\partial L'}=-2\pi\nu_1'' \qquad \frac{\partial F_0^{**}}{\partial G''}=2\pi(\nu_1''-\nu_2'') \qquad \frac{\partial F_0^{**}}{\partial H}=2\pi(\nu_2''\operatorname{sgn}\alpha_3-\nu_3''). \tag{9.00}$$

From (8.10) and (6.12), with use of $e'^2=1-G''^2/L'^2$, $L'^2=\mu a'$, and $\eta_0''^2=1-H^2/G''^2$, we have with sufficient accuracy

$$F_1^{**}=-\frac{3\mu^6 r_e^4\sigma_4}{128L'^3G''^7}\left(5-3\frac{G''^2}{L'^2}\right)\left(3-30\frac{H^2}{G''^2}+35\frac{H^4}{G''^4}\right). \tag{9.01}$$

Then

$$\frac{\partial F_1^{**}}{\partial L'}=\frac{45\mu^6 r_e^4\sigma_4}{128L'^4G''^7}\left(1-\frac{G''^2}{L'^2}\right)\left(3-30\frac{H^2}{G''^2}+35\frac{H^4}{G''^4}\right) \tag{9.02}$$

$$\frac{\partial F_1^{**}}{\partial G''}=\frac{15\mu^6 r_e^4\sigma_4}{128L'^3G''^8}\left[3\left(7-3\frac{G''^2}{L'^2}\right)+18\left(7\frac{G''^2}{L'^2}-15\right)\frac{H^2}{G''^2}+7\left(55-27\frac{G''^2}{L'^2}\right)\frac{H^4}{G''^4}\right] \tag{9.03}$$

$$\frac{\partial F_1^{**}}{\partial H}=\frac{15\mu^6 r_e^4\sigma_4}{32L'^3G''^9}H\left(5-3\frac{G''^2}{L'^2}\right)\left(3-7\frac{H^2}{G''^2}\right). \tag{9.04}$$

Since our constant orbital elements $a$, $e$, and $I$, and thus the frequencies $\nu_1$ and $\nu_2$, correspond to $L'$, $G''$, and $H$, we may drop the double primes from the $\nu$'s in (9.00). Also, to our specified accuracy, we may put $e^2=1-G''^2/L'^2$, $L'^2=\mu a$, and $\eta_0^2=1-H^2/G''^2$ in (9.02) through (9.04). Then,

with the unperturbed values $l$, $g$, and $h$ given by (4.18), we find

$$l'' = l + \sigma_4 l_{42} \qquad g'' = g + \sigma_4 g_{42} \qquad h'' = h + \sigma_4 h_{42}, \tag{9.05}$$

where

$$l_{42} = -\frac{45}{128}\left(\frac{r_e}{p}\right)^4 n(1-e^2)^{1/2}(3 - 30\cos^2 I + 35\cos^4 I)t \tag{9.06}$$

$$g_{42} = -\frac{15}{128}\left(\frac{r_e}{p}\right)^4 n[3(4+3e^2) - 18(8+7e^2)\cos^2 I + 7(28+27e^2)\cos^4 I]t \tag{9.07}$$

$$h_{42} = -\frac{15}{32}\left(\frac{r_e}{p}\right)^4 n(2+3e^2)\cos I(3 - 7\cos^2 I)t. \tag{9.08}$$

We postpone summarizing the results for the residual fourth harmonic until the complete algorithm in section 11.

## 10. Effects of the Third Zonal Harmonic

By (2.01) the contribution of the third zonal harmonic to the potential is

$$\Delta V = \mu r_e^3 r^{-4} J_3 P_3(\sin\theta). \tag{10.00}$$

Corresponding to (3.02), this leads to

$$\Delta F_1 = -\mu r_e^3 r^{-4} J_3 P_3(\sin\theta) \tag{10.01}$$

and thus to

$$\Delta F_1 = -\frac{\mu r_e^3}{a^4} J_3\left[-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3\left(\frac{a}{r}\right)^4 \sin(v+g) - \frac{5}{8}\eta_0^3\left(\frac{a}{r}\right)^4 \sin(3v+3g)\right], \tag{10.02}$$

corresponding to (6.06). Following the notation of section 6, we then find

$$\Delta F_{1m} = (2\pi)^{-1}\int_0^{2\pi} \Delta F_1 dl = -\frac{\mu r_e^3}{a^4} J_3 e(1-e^2)^{-5/2}\left(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3\right)\sin g, \tag{10.03}$$

which is purely long-periodic, so that

$$\Delta F_{1m} = \Delta F_{1p} \qquad (10.04) \qquad \Delta F_{1c} = 0. \tag{10.05}$$

Since $\Delta F_{1c} = 0$, there are no secular effects.

The short-periodic part of $\Delta F_1$ is then

$$\Delta F_{1l} = \Delta F_1 - \Delta F_{1m} = -\frac{\mu r_e^3}{a^4} J_3\left[\left(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3\right)\left\{\left(\frac{a}{r}\right)^4 \sin(v+g) - e(1-e^2)^{-5/2}\sin g\right\} - \frac{5}{8}\eta_0^3\left(\frac{a}{r}\right)^4 \sin(3v+3g)\right]. \tag{10.06}$$

On following the same procedure as in section 6, we find

$$\frac{\partial \Delta S_1}{\partial l} = -(2\pi\nu_1)^{-1}\Delta F_{1l} \tag{10.07}$$

where

$$\Delta S_1 = -(2\pi\nu_1)^{-1}\int \Delta F_{1l}\,dl \tag{10.08}$$

$$= -\frac{\mu r_e^3}{96na^4} J_3(1-e^2)^{-5/2}\eta_0\{3(1-5\cos^2 I)[12e(v-l)\sin g + 3e^2\cos(v-g)$$

$$-6(2+e^2)\cos(v+g) - 6e\cos(2v+g) - e^2\cos(3v+g)] + \eta_0^2[15e^2\cos(v+3g) + 30e\cos(2v+3g)$$

$$+10(2+e^2)\cos(3v+3g) + 15e\cos(4v+3g) + 3e^2\cos(5v+3g)]\}. \tag{10.09}$$

Before taking derivatives of (10.09) with respect to $L'$ and $G'$ one must replace $e^2$ by $1-G'^2/L'^2$, $\eta_0^2$ by $1-H^2/G'^2$, and $a$ by $L'^2/\mu$. The short-periodic effects are then given by

$$\frac{\partial \Delta S_1}{\partial l} = L - L' = J_3 L_3 \qquad -\frac{\partial \Delta S_1}{\partial L'} = l - l' = J_3 l_3$$

$$\frac{\partial \Delta S_1}{\partial g} = G - G' = J_3 G_3 \tag{10.10} \qquad -\frac{\partial \Delta S_1}{\partial G'} = g - g' = J_3 g_3 \tag{10.11}$$

$$\frac{\partial \Delta S_1}{\partial h} = H - H' = 0 \qquad -\frac{\partial \Delta S_1}{\partial H} = h - h' = J_3 h_3,$$

where

$$L_3 = -\frac{nr_e^3}{a}\left\{\left(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3\right)\left[\left(\frac{a}{r}\right)^4 \sin(v+g) - e(1-e^2)^{-5/2}\sin g\right]\right.$$

$$\left. -\frac{5}{8}\eta_0^3\left(\frac{a}{r}\right)^4 \sin(3v+3g)\right]\Big\}. \tag{10.12}$$

$$G_3 = -na^3(1-e^2)^{1/2}\left(\frac{r_e}{p}\right)^3\left\{\left(-\frac{3}{2}\eta_0 + \frac{15}{8}\eta_0^3\right)\left[e(v-l)\cos g + \frac{e^2}{4}\sin(v-g)\right.\right.$$

$$\left. + \left(1+\frac{1}{2}e^2\right)\sin(v+g) + \frac{1}{2}e\sin(2v+g) + \frac{e^2}{12}\sin(3v+g)\right] - \frac{15}{8}\eta_0^3\left[\frac{e^2}{4}\sin(v+3g) + \frac{1}{2}e\sin(2v+3g)\right.$$

$$\left.\left. + \frac{1}{3}\left(1+\frac{1}{2}e^2\right)\sin(3v+3g) + \frac{e}{4}\sin(4v+3g) + \frac{e^2}{20}\sin(5v+3g)\right]\right\}. \tag{10.13}$$

$$l_3 = \frac{\eta_0}{8e}(1-e^2)^{1/2}\left(\frac{r_e}{p}\right)^3\left\{3(5\eta_0^2-4)\left[(1-e^2)(v-l)\sin g + \left(1+\frac{1}{2}e^2\right)\cos g\right.\right.$$

$$+\frac{e}{8}(14-3e^2)\cos(v-g) + \frac{1}{2}e^2\cos(2v-g) + \frac{e^3}{16}\cos(3v-g) - e(1-e^2)\cos(v+g)$$

$$\left. -\frac{3}{2}\cos(2v+g) - \frac{e}{24}(34-e^2)\cos(3v+g) - \frac{1}{2}e^2\cos(4v+g) - \frac{e^3}{16}\cos(5v+g)\right]$$

$$+5\eta_0^2\left[-\frac{1}{2}e^2\cos 3g - \frac{e^3}{16}\cos(v-3g) - \frac{e}{8}(6+5e^2)\cos(v+3g) - \left(\frac{1}{2}+e^2\right)\cos(2v+3g)\right.$$

$$+\frac{e}{3}(1-e^2)\cos(3v+3g) + \frac{1}{4}(5+e^2)\cos(4v+3g) + \frac{e}{40}(54+e^2)\cos(5v+3g)$$

$$\left.\left. + \frac{1}{2}e^2\cos(6v+3g) + \frac{e^3}{16}\cos(7v+3g)\right]\right\}. \tag{10.14}$$

$$g_3 = -(1-e^2)^{-1/2}l_3 - \frac{1}{8}\left(\frac{r_e}{p}\right)^3\left\{\eta_0^{-1}(4-39\eta_0^2+40\eta_0^4)\left[e(v-l)\sin g + \frac{e^4}{4}\cos(v-g)\right.\right.$$

$$\left.\left. -\left(1+\frac{1}{2}e^2\right)\cos(v+g) - \frac{1}{2}e\cos(2v+g) - \frac{e^2}{12}\cos(3v+g)\right] + 5\eta_0(8\eta_0^2-3)\left[\frac{e^2}{4}\cos(v+3g)\right.\right.$$

$$+\frac{1}{2}e\cos(2v+3g)+\frac{1}{3}\left(1+\frac{1}{2}e^2\right)\cos(3v+3g)+\frac{e}{4}\cos(4v+3g)+\frac{e^2}{20}\cos(5v+3g)\Big]\Big\}. \tag{10.15}$$

$$h_3=\left(\frac{r_e}{p}\right)^3\left\{\frac{3}{2}\cot I\left(1-\frac{15}{4}\sin^2 I\right)\left[e(v-l)\sin g+\frac{e^2}{4}\cos(v-g)-\left(1+\frac{1}{2}e^2\right)\cos(v+g)\right.\right.$$

$$\left.-\frac{1}{2}e\cos(2v+2g)-\frac{e^2}{12}\cos(3v+2g)\right]-\frac{15}{16}\sin 2I\left[\frac{e^2}{4}\cos(v+3g)\right.$$

$$\left.\left.+\frac{1}{2}e\cos(2v+3g)+\frac{1}{3}\left(1+\frac{1}{2}e^2\right)\cos(3v+3g)+\frac{e}{4}\cos(4v+3g)+\frac{e^2}{20}\cos(5v+3g)\right]\right\}. \tag{10.16}$$

Continuing on to the long-periodic effects and following the procedures of section 8, we find

$$(\Delta F_1)^{**}=0, \tag{10.17}$$

so that the third harmonic gives no secular changes, and

$$2\pi(\nu_1''-\nu_2'')\frac{\partial\Delta S_1^*}{\partial g'}=-\Delta F_{1p}. \tag{10.18}$$

Now

$$F_{1p}=-\frac{3}{8}\frac{\mu^5 r_e^3 e''}{L'^3 G''^5}J_3\eta_0''\left(1-5\frac{H^2}{G''^2}\right)\sin g', \tag{10.19}$$

by (10.03) and (10.04). With use of (10.18) and (10.19) and of (8.13) for $2\pi(\nu_1''-\nu_2'')$, we find

$$\frac{\partial\Delta S_1^*}{\partial g'}=\frac{1}{2}\frac{\mu r_e}{G''}\frac{J_3}{J_2}e''\eta_0''\sin g', \tag{10.20}$$

so that

$$\Delta S_1^*=-\frac{1}{2}\frac{\mu r_e}{G''}\frac{J_3}{J_2}e''\eta_0''\cos g'. \tag{10.21}$$

It follows that

$$\begin{aligned}
&\frac{\partial\Delta S_1^*}{\partial l'}=L'-L''=0 && -\frac{\partial\Delta S_1^*}{\partial L'}=l'-l''=J_3\tilde{l}_3/J_2\\
&\frac{\partial\Delta S_1^*}{\partial g'}=G'-G''=J_3\tilde{G}_3/J_2 \quad (10.22) && -\frac{\partial\Delta S_1^*}{\partial G'}=g'-g''=J_3\tilde{g}_3/J_2 \quad (10.23)\\
&\frac{\partial\Delta S_1^*}{\partial h'}=H'-H''=0 && -\frac{\partial\Delta S_1^*}{\partial H'}=h'-h''=J_3\tilde{h}_3/J_2,
\end{aligned}$$

where

$$\tilde{G}_3=\frac{1}{2}r_e na(1-e^2)^{-\frac{1}{2}}e\eta_0\sin g \tag{10.24}$$

$$\tilde{l}_3=\frac{1}{2}\frac{r_e}{a}\frac{(1-e^2)^{\frac{1}{2}}}{e}\sin I\cos g \tag{10.25}$$

$$\tilde{g}_3=\frac{1}{2}\frac{r_e}{p}\left(\frac{e\cos^2 I}{\sin I}-\frac{\sin I}{e}\right)\cos g \tag{10.26}$$

$$\tilde{h}_3=-\frac{1}{2}\frac{r_e}{p}e\cot I\cos g. \tag{10.27}$$

A summary of the results for the third harmonic will appear in the next section, which gives the complete algorithm.

## 11. The Complete Algorithm

We shall summarize results by writing out the complete algorithm for the calculation of the motion in the potential field (1.00), as modified by the third harmonic and the residual fourth harmonic. This will involve a repetition, with some changes, of about two pages of an earlier paper [Vinti 1961 a], but it is highly desirable to assemble the whole solution in one place.

Let the planetary constants $\mu$, $r_e$, $J_2$, $J_3$, and $J_4$ be given, along with the constant orbital elements $a$, $e$, $I$, $l_0$, $g_0$, and $\beta_3$. To calculate the

### Unperturbed Reference Orbit

compute

$$c = r_e J_2^{1/2} \qquad \eta_0 = \sin I \qquad p = a(1-e^2)$$

$$D = (ap - c^2)(ap - c^2\eta_0^2) + 4a^2c^2\eta_0^2 \qquad D' = 4a^2c^2(1-\eta_0^2) + D$$

$$A = -2ac^2D^{-1}(1-\eta_0^2)(ap - c^2\eta_0^2) \qquad B = c^2\eta_0^2 D^{-1}D'$$

$$b_1 = -\frac{1}{2}A \qquad b_2 = B^{1/2} \qquad -2\alpha_1 = \mu(a+b_1)^{-1}$$

$$-\frac{\alpha_2^2}{2\alpha_1} = a_0p_0 = -c^2(1-\eta_0^2) + apD^{-1}D' \qquad \alpha_2 = (-2\alpha_1)^{1/2}(a_0p_0)^{1/2} > 0$$

$$\alpha_3 = \alpha_2\left(1 - \frac{c^2\eta_0^2}{a_0p_0}\right)^{1/2}\cos I \qquad \eta_2^{-2} = \frac{c^2D}{apD'} \qquad k = c^2p^{-2} \qquad q = \eta_0\eta_2^{-1}$$

$$A_1 = (1-e^2)^{1/2}\, p \sum_{n=2}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_{n-2}[(1-e^2)^{1/2}]$$

$$A_2 = (1-e^2)^{1/2}\, p^{-1} \sum_{n=0}^{\infty} (b_2/p)^n P_n(b_1/b_2) R_n[(1-e^2)^{1/2}],$$

where $P_n(x)$ is the Legendre polynomial of degree $n$ and where $R_n(x) \equiv x^n P_n(x^{-1})$, always a polynomial of degree $[n/2]$ in $x^2$.

$$A_3 = (1-e^2)^{1/2}\, p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2}[(1-e^2)^{1/2}],$$

where

$$D_{2i} = \sum_{n=0}^{i} (-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n} P_{2n}(b_1/b_2)$$

$$D_{2i+1} = \sum_{n=0}^{i} (-1)^{i-n}(c/p)^{2i-2n}(b_2/p)^{2n+1} P_{2n+1}(b_1/b_2)$$

$$B_1 = 2\pi^{-1}q^{-2}[K(q) - E(q)] \qquad B_2 = 2\pi^{-1}K(q),$$

where $K(q)$ and $E(q)$ are the complete elliptic integrals of the first and second kinds, respectively. It may be convenient to have the series

$$B_1 = \frac{1}{2} + \frac{3}{16}q^2 + \frac{15}{128}q^4 + \ldots \qquad B_2 = 1 + \frac{1}{4}q^2 + \frac{9}{64}q^4 + \ldots$$

Continue with

$$B_3 = 1-(1-\eta_2^{-2})^{-\frac{1}{2}} - \sum_{m=2}^{\infty} \gamma_m \eta_2^{-2m}, \text{ where } \gamma_m = \frac{(2m)!}{2^{2m}(m!)^2} \sum_{n=1}^{m-1} \frac{(2n)!\eta_0^{2n}}{2^{2n}(n!)^2}$$

$$A_{11} = \frac{3}{4}(1-e^2)^{\frac{1}{2}} p^{-3} e(-2b_1 b_2^2 p + b_2^4) \qquad A_{12} = \frac{3}{32}(1-e^2)^{\frac{1}{2}} p^{-3} b_2^4 e^2$$

$$A_{21} = (1-e^2)^{\frac{1}{2}} p^{-1} e\left[b_1 p^{-1} + (3b_1^2 - b_2^2)p^{-2} - \frac{9}{2} b_1 b_2^2\left(1+\frac{e^2}{4}\right)p^{-3} + \frac{3}{8} b_2^4(4+3e^2)p^{-4}\right]$$

$$A_{22} = (1-e^2)^{\frac{1}{2}} p^{-1}\left[\frac{e^2}{8}(3b_1^2 - b_2^2)p^{-2} - \frac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3}{32} b_2^4(6e^2 + e^4)p^{-4}\right]$$

$$A_{23} = (1-e^2)^{\frac{1}{2}} p^{-1} \frac{e^3}{8}(-b_1 b_2^2 p^{-3} + b_2^4 p^{-4})$$

$$A_{24} = \frac{3}{256}(1-e^2)^{\frac{1}{2}} p^{-5} b_2^4 e^4$$

$$A_{31} = (1-e^2)^{\frac{1}{2}} p^{-3} e\left[2 + b_1 p^{-1}\left(3+\frac{3e^2}{4}\right) - p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right)(4+3e^2)\right]$$

$$A_{32} = (1-e^2)^{\frac{1}{2}} p^{-3}\left[\frac{e^2}{4} + \frac{3}{4} b_1 p^{-1} e^2 - p^{-2}\left(\frac{e^2}{4} + \frac{3}{2} e^2\right)\left(\frac{1}{2} b_2^2 + c^2\right)\right]$$

$$A_{33} = (1-e^2)^{\frac{1}{2}} p^{-3} e^3\left[\frac{b_1}{12} p^{-1} - \frac{1}{3} p^{-2}\left(\frac{1}{2} b_2^2 + c^2\right)\right]$$

$$A_{34} = -\frac{1}{32}(1-e^2)^{\frac{1}{2}} p^{-5} e^4\left(\frac{1}{2} b_2^2 + c^2\right)$$

$$2\pi\nu_1 = (-2\alpha_1)^{\frac{1}{2}}(a + b_1 + A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1})^{-1}$$

$$2\pi\nu_2 = (\alpha_2^2 - \alpha_3^2)^{\frac{1}{2}} \eta_0^{-1} A_2 B_2^{-1}(a + b_1 + A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1})^{-1}.$$

The uniformising variables $E$, $v$, and $\psi$ are then given by $E = M_s + E_p$, $v = M_s + E_p$, and $\psi = \psi_s + \psi_p$. If $t$ is the time, their secular parts $M_s$ and $\psi_s$ are given exactly by

$$M_s = l_0 + 2\pi\nu_1 t \qquad \psi_s = l_0 + g_0 + 2\pi\nu_2 t.$$

Let the periodic parts be split as follows: $E_p = E_0 + E_1 + E_2$, $v_p = v_0 + v_1 + v_2$, and $\psi_p = \psi_0 + \psi_1 + \psi_2$, where, e.g., $E_0$ contains terms of order $J_2^0$, $J_2$, and $J_2^2$, $E_1$ contains terms of order $J_2$ and $J_2^2$, and $E_2$ contains terms of order $J_2^2$ only.

Then $E_0$ is given by the Kepler equation

$$M_s + E_0 - e' \sin(M_s + E_0) = M_s,$$

where $e' \equiv ae(a+b_1)^{-1} < e$. The term $v_0$ is then given by placing $v = M_s + v_0$ and $E = M_s + v_0$ in the *anomaly connections*

$$\cos v = (\cos E - e)(1 - e\cos E)^{-1} \qquad \sin v = (1-e^2)^{\frac{1}{2}}(1 - e\cos E)^{-1} \sin E$$

or equivalent relations. (Note that $e$ here is the original $e$ and not the $e'$ in the Kepler equation.) Then

$$\psi_0 = (-2\alpha_1)^{-\frac{1}{2}}(\alpha_2^2 - \alpha_3^2)^{\frac{1}{2}} \eta_0^{-1} A_2 B_2^{-1} v_0.$$

The term $E_1$ is now given by

$$E_1 = [1 - e'\cos(M_s + E_0)]^{-1} M_1 - \frac{1}{2} e'[1 - e'\cos(M_s + E_0)]^{-3} M_1^2 \sin(M_s + E_0),$$

where

$$M_1=(a+b_1)^{-1}\left[-(A_1+c^2\eta_0^2A_2B_1B_2^{-1})v_0+\frac{c^2}{4}(-2\alpha_1)^{-1/2}\,(\alpha_2^2-\alpha_3^2)^{-1/2}\,\eta_0^3\sin(2\psi_s+2\psi_0)\right].$$

The term $v_1$ is then given by placing $v=M_s+v_0+v_1$ and $E=M_s+E_0+E_1$ in the *anomaly connections*. Then

$$\psi_1=(-2\alpha_1)^{-1/2}\,(\alpha_2^2-\alpha_3^2)^{1/2}\,\eta_0^{-1}B_2^{-1}[A_2v_1+A_{21}\sin(M_s+v_0)+A_{22}\sin(2M_s+2v_0)]$$
$$+\frac{q^2}{8}B_2^{-1}\sin(2\psi_s+2\psi_0).$$

Finally

$$E_2=[1-e'\cos(M_s+E_0+E_1)]^{-1}M_2,$$

where

$$M_2=-(a+b_1)^{-1}\left[A_1v_1+A_{11}\sin(M_s+v_0)+A_{12}\sin(2M_s+2v_0)\right.$$
$$\left.+c^2(-2\alpha_1)^{1/2}(\alpha_2^2-\alpha_3^2)^{-1/2}\,\eta_0^3\left\{B_1\psi_1-\tfrac{1}{2}\psi_1\cos(2\psi_s+2\psi_0)-\frac{q^2}{8}\sin(2\psi_s+2\psi_0)+\frac{q^2}{64}\sin(4\psi_s+4\psi_0)\right\}\right].$$

Then $v_2$ is found by placing $v=M_s+v_0+v_1+v_2$ and $E=M_s+E_0+E_1+E_2$ in the *anomaly connections* and

$$\psi_2=(-2\alpha_1)^{-1/2}\,(\alpha_2^2-\alpha_3^2)^{1/2}\,\eta_0^{-1}B_2^{-1}[A_2v_2+A_{21}v_1\cos(M_s+v_0)+2A_{22}v_1\cos(2M_s+2v_0)$$
$$+A_{23}\sin(3M_s+3v_0)+A_{24}\sin(4M_s+4v_0)]+\frac{q^2}{4}B_2^{-1}\left[\psi_1\cos(2\psi_s+2\psi_0)\right.$$
$$\left.+\frac{3q^2}{8}\sin(2\psi_s+2\psi_0)-\frac{3q^2}{64}\sin(4\psi_s+4\psi_0)\right].$$

## Unperturbed Reference Orbit

The spheroidal coordinates $\rho$ and $\eta$ are then given by

$$\rho=a(1-e\cos E)=(1+e\cos v)^{-1}p \qquad \eta=\eta_0\sin\psi,$$

where $E=M_s+E_0+E_1+E_2$ and $v=M_s+v_0+v_1+v_2$. To obtain the *unperturbed right ascension* $\phi$, first calculate an angle $\chi$, equaling $\psi$ whenever $\psi$ is a multiple of $\pi/2$ and satisfying

$$\cos\chi=(1-\eta_0^2\sin^2\psi)^{-1/2}\cos\psi \qquad \sin\chi=(1-\eta_0^2)^{1/2}\,(1-\eta_0^2\sin^2\psi)^{-1/2}\sin\psi$$

Then

$$\phi=\beta_3+\alpha_3(\alpha_2^2-\alpha_3^2)^{-1/2}\,\eta_0\left[(1-\eta_0^2)^{-1/2}\,(1-\eta_2^{-2})^{-1/2}\chi+B_3\psi+\frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]$$
$$-c^2\alpha_3(-2\alpha_1)^{-1/2}\left[A_3v+\sum_{n=1}^{4}A_{3n}\sin nv\right].$$

## The Zonal Harmonic Perturbations

Within the accuracy of calculation of the perturbations, we may use either the approximate or the accurate formulas listed below. (Use of the approximate formulas may involve more work than

use of the accurate ones, because the latter will already be known from the solution of the unperturbed problem.) Compute

| | Approximate formula | Accurate formula |
|---|---|---|
| $n$ | $\mu^{1/2}a^{-3/2}$ | |
| $l$ | $l_0+nt$ | $M_s=l_0+2\pi\nu_1 t$ |
| $g$ | $g_0+\frac{3}{4}\left(\frac{r_e}{p}\right)^2 nJ_2(5\cos^2 I-1)t$ | $g_0+2\pi(\nu_2-\nu_1)t$ |
| $E$ | $M_s+E_0$ | $M_s+E_0+E_1+E_2$ |
| $v$ | $M_s+v_0$ | $M_s+v_0+v_1+v_2$ |

Also, $r=\rho$, within the accuracy of the calculation. Then compute

### Third Harmonic Short-Periodic Quantities

$$L_3=-\frac{nr_e^3}{a}\left[\left(-\frac{3}{2}\eta_0+\frac{15}{8}\eta_0^3\right)\left\{\left(\frac{a}{r}\right)^4\sin(v+g)-e(1-e^2)^{-5/2}\sin g\right\}-\frac{5}{8}\eta_0^3\left(\frac{a}{r}\right)^4\sin(3v+3g)\right]$$

$$\begin{aligned}G_3=&-na^2(1-e^2)^{1/2}\left(\frac{r_e}{p}\right)^3\left\{\left(-\frac{3}{2}\eta_0+\frac{15}{8}\eta_0^3\right)\left[e(v-l)\cos g+\frac{e^2}{4}\sin(v-g)\right.\right.\\&+\left(1+\frac{1}{2}e^2\right)\sin(v+g)+\frac{1}{2}e\sin(2v+g)+\frac{e^2}{12}\sin(3v+g)\Big]\\&-\frac{15}{8}\eta_0^3\left[\frac{e^2}{4}\sin(v+3g)+\frac{1}{2}e\sin(2v+3g)+\frac{1}{3}\left(1+\frac{1}{2}e^2\right)\sin(3v+3g)\right.\\&\left.\left.+\frac{e}{4}\sin(4v+3g)+\frac{e^2}{20}\sin(5v+3g)\right]\right\}.\end{aligned}$$

(Note that $v-l=v_0$, within the accuracy of the calculation.)

$$\begin{aligned}l_3=&\frac{3}{8e}(1-e^2)^{1/2}\left(\frac{r_e}{p}\right)^3\eta_0(5\eta_0^2-4)\left[(1-e^2)(v-l)\sin g+\left(1+\frac{1}{2}e^2\right)\cos g+\frac{e}{8}(14-3e^2)\cos(v-g)\right.\\&+\frac{1}{2}e^2\cos(2v-g)+\frac{e^3}{16}\cos(3v-g)-e(1-e^2)\cos(v+g)-\frac{3}{2}\cos(2v+g)-\frac{e}{24}(34-e^2)\cos(3v+g)\\&\left.-\frac{1}{2}e^2\cos(4v+g)-\frac{e^3}{16}\cos(5v+g)\right]+\frac{5}{8e}(1-e^2)^{1/2}\left(\frac{r_e}{p}\right)^3\eta_0^3\left[-\frac{1}{2}e^2\cos 3g-\frac{e^3}{16}\cos(v-3g)\right.\\&-\frac{e}{8}(6+5e^2)\cos(v+3g)-\left(\frac{1}{2}+e^2\right)\cos(2v+3g)+\frac{e}{3}(1-e^2)\cos(3v+3g)+\frac{1}{4}(5+e^2)\cos(4v+3g)\\&\left.+\frac{e}{40}(54+e^2)\cos(5v+3g)+\frac{1}{2}e^2\cos(6v+3g)+\frac{e^3}{16}\cos(7v+3g)\right]\end{aligned}$$

$$\begin{aligned}g_3=&-(1-e^2)^{-1/2}l_3-\frac{3}{8}\left(\frac{r_e}{p}\right)^3\eta_0^{-1}(4-39\eta_0^2+40\eta_0^4)\left[e(v-l)\sin g+\frac{e^2}{4}\cos(v-g)\right.\\&\left.-\left(1+\frac{1}{2}e^2\right)\cos(v+g)-\frac{1}{2}e\cos(2v+g)-\frac{e^2}{12}\cos(3v+g)\right]\\&-\frac{5}{8}\left(\frac{r_e}{p}\right)^3\eta_0(8\eta_0^2-3)\left[\frac{e^2}{4}\cos(v+3g)+\frac{1}{2}e\cos(2v+3g)+\frac{1}{3}\left(1+\frac{1}{2}e^2\right)\cos(3v+3g)\right.\\&\left.+\frac{e}{4}\cos(4v+3g)+\frac{e^2}{20}\cos(5v+3g)\right]\end{aligned}$$

$$h_3=\left(\frac{r_e}{p}\right)^3\left\{\frac{3}{2}\cot I\left(1-\frac{15}{4}\eta_0^2\right)\left[e(v-l)\sin g+\frac{e^2}{4}\cos(v-g)-\left(1+\frac{1}{2}e^2\right)\cos(v+g)\right.\right.$$
$$-\frac{1}{2}e\cos(2v+g)-\frac{e^2}{12}\cos(3v+g)\Big]-\frac{15}{16}\sin 2I\left[\frac{e^2}{4}\cos(v+3g)+\frac{1}{2}e\cos(2v+3g)\right.$$
$$\left.\left.+\frac{1}{3}\left(1+\frac{1}{2}e^2\right)\cos(3v+3g)+\frac{e}{4}\cos(4v+3g)+\frac{e^2}{20}\cos(5v+3g)\right]\right\}.$$

### Third Harmonic Long-Periodic Quantities

$$\tilde{G}_3=\frac{1}{2}\dot{r}_e na(1-e^2)^{-1/2}e\eta_0\sin g$$

$$\tilde{l}_3=\frac{r_e}{2ae}(1-e^2)^{1/2}\eta_0\cos g$$

$$\tilde{g}_3=\frac{r_e}{2p}\left(\frac{e\cos^2 I}{\sin I}-\frac{\sin I}{e}\right)\cos g$$

$$\tilde{h}_3=-\frac{r_e}{2p}e\cot I\cos g$$

### Fourth Harmonic Short-Periodic Quantities

$$L_4=-\frac{1}{8}\left(\frac{r_e}{a}\right)^4(\mu a)^{1/2}\left\{\left(3-15\eta_0^2+\frac{105}{8}\eta_0^4\right)\left[\left(\frac{a}{r}\right)^5-(1-e^2)^{-7/2}\left(1+\frac{3}{2}e^2\right)\right]\right.$$
$$+5\eta_0^2\left(3-\frac{7}{2}\eta_0^2\right)\left[\left(\frac{a}{r}\right)^5\cos(2v+2g)-\frac{3}{4}e^2(1-e^2)^{-7/2}\cos 2g\right]$$
$$\left.+\frac{35}{8}\eta_0^4\left(\frac{a}{r}\right)^5\cos(4v+4g)\right\}$$

$$G_4=-\frac{5}{8}\left(\frac{r_e}{p}\right)^4(\mu p)^{1/2}\eta_0^2\left\{\left(3-\frac{7}{2}\eta_0^2\right)\left[-\frac{3}{2}e^2(v-l)\sin 2g-\frac{e^3}{4}\cos(v-2g)\right.\right.$$
$$+\left(3e+\frac{3e^3}{4}\right)\cos(v+2g)+\left(1+\frac{3}{2}e^2\right)\cos(2v+2g)+\left(e+\frac{e^3}{4}\right)\cos(3v+2g)$$
$$+\frac{3e^2}{8}\cos(4v+2g)+\frac{e^3}{20}\cos(5v+2g)\Big]+\frac{7}{8}\eta_0^2\left[\frac{1}{2}e^3\cos(v+4g)+\frac{3e^2}{2}\cos(2v+4g)\right.$$
$$+\left(2e+\frac{1}{2}e^3\right)\cos(3v+4g)+\left(1+\frac{3}{2}e^2\right)\cos(4v+4g)+\left(\frac{6e}{5}+\frac{3e^3}{10}\right)\cos(5v+4g)$$
$$\left.\left.+\frac{1}{2}e^2\cos(6v+4g)+\frac{e^3}{14}\cos(7v+4g)\right]\right\}$$

$$l_{41}=\left(\frac{r_e}{p}\right)^4\frac{(1-e^2)^{1/2}}{2048}\left\{6(8-40\eta_0^2+35\eta_0^4)\left[48(1-e^2)(v-l)\right.\right.$$
$$+\frac{2}{e}(40+12e^2-17e^4)\sin v+4(20-e^2)\sin 2v+e(40-e^2)\sin 3v+10e^2\sin 4v$$
$$\left.+e^3\sin 5v\right]-4\eta_0^2(7\eta_0^2-6)\left[240(1-e^2)(v-l)\cos 2g+5e^3\sin(3v-2g)\right.$$
$$+50e^2\sin(2v-2g)+15e(16-3e^2)\sin(v-2g)-20(14+5e^2)\sin 2g$$
$$+\frac{10}{e}(8-24e^2-19e^4)\sin(v+2g)+240(1-e^2)\sin(2v+2g)+\frac{10}{e}(24+16e^2-5e^4)\sin(3v+2g)$$
$$\left.\left.+20(17+2e^2)\sin(4v+2g)+3e(64+e^2)\sin(5v+2g)+50e^2\sin(6v+2g)+5e^3\sin(7v+2g)\right]\right.$$

$$+\eta_0^4\Big[35e^3\sin(v-4g)-350e^2\sin 4g-105e(8+5e^2)\sin(v+4g)-140(8+11e^2)\sin(2v+4g)$$
$$-\frac{70}{e}(8+20e^2+7e^4)\sin(3v+4g)+840(1-e^2)\sin(4v+4g)+\frac{14}{e}(104+84e^2-13e^4)\sin(5v+4g)$$
$$+140(16+3e^2)\sin(6v+4g)+15e(88+3e^2)\sin(7v+4g)+350e^2\sin(8v+4g)$$
$$+35e^3\sin(9v+4g)\Big]\Big\}$$

$$g_{41}=-(1-e^2)^{-7/2}l_{41}-\frac{(r_e/p)^4}{512}\Big\{2(136-500\eta_0^2+385\eta_0^4)[6(2+3e^2)(v-l)$$
$$+9e(4+e^2)\sin v+9e^2\sin 2v+e^3\sin 3v]-2(12-82\eta_0^2+77\eta_0^4)[60e^2(v-l)\cos 2g$$
$$+10e^3\sin(v-2g)+30e(4+e^2)\sin(v+2g)+20(2+3e^2)\sin(2v+2g)+10e(4+e^2)\sin(3v+2g)$$
$$+15e^2\sin(4v+2g)+2e^3\sin(5v+2g)]+\eta_0^2(11\eta_0^2-4)[35e^3\sin(v+4g)+105e^2\sin(2v+4g)$$
$$+35e(4+e^2)\sin(3v+4g)+35(2+3e^2)\sin(4v+4g)+21e(4+3e^2)\sin(5v+4g)$$
$$+35e^2\sin(6v+4g)+5e^3\sin(7v+4g)]\Big\}$$

$$h_{41}=-\frac{5}{16}\left(\frac{r_e}{p}\right)^4\cos I\Big\{3(7\eta_0^2-4)\Big[\Big(1+\frac{3}{2}e^2\Big)(v-l)+\Big(3e+\frac{3}{4}e^3\Big)\sin v+\frac{3}{4}e^2\sin 2v$$
$$+\frac{e^3}{12}\sin 3v\Big]+4(3-7\eta_0^2)\Big[\frac{3}{4}e^2(v-l)\cos 2g+\frac{e^3}{8}\sin(v-2g)+\Big(\frac{3e}{2}+\frac{3e^3}{8}\Big)\sin(v+2g)$$
$$+\Big(\frac{1}{2}+\frac{3e^2}{4}\Big)\sin(2v+2g)+\Big(\frac{1}{2}e+\frac{e^3}{8}\Big)\sin(3v+2g)+\frac{3e^2}{16}\sin(4v+2g)+\frac{e^3}{40}\sin(5v+2g)\Big]$$
$$+7\eta_0^2\Big[\frac{e^3}{8}\sin(v+4g)+\frac{3e^2}{8}\sin(2v+4g)+\Big(\frac{1}{2}e+\frac{e^3}{8}\Big)\sin(3v+4g)+\Big(\frac{1}{4}+\frac{3e^2}{8}\Big)\sin(4v+4g)$$
$$+\Big(\frac{3e}{10}+\frac{3e^3}{40}\Big)\sin(5v+4g)+\frac{e^2}{8}\sin(6v+4g)+\frac{e^3}{56}\sin(7v+4g)\Big]\Big\}$$

### Fourth Harmonic Long-Periodic Quantities

$$\tilde{G}_4=-\frac{5r_e^2n}{16}e^2(1-e^2)^{-3/2}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\cos 2g$$

$$\tilde{l}_4=\frac{5}{16}\left(\frac{r_e}{a}\right)^2(1-e^2)^{-7/2}\eta_0^2(1-7\cos^2 I)(1-5\cos^2 I)^{-1}\sin 2g$$

$$\tilde{g}_4=-\frac{5}{32}\left(\frac{r_e}{p}\right)^2[2+e^2-3(2+3e^2)\cos^2 I-8(2+5e^2)\cos^4 I(1-5\cos^2 I)^{-1}$$
$$-80e^2\cos^6 I(1-5\cos^2 I)^{-2}]\sin 2g$$

$$\tilde{h}_4=-\frac{5}{16}\left(\frac{r_e}{p}\right)^2e^2\cos I[3+16\cos^2 I(1-5\cos^2 I)^{-1}+40\cos^4 I(1-5\cos^2 I)^{-2}]\sin 2g.$$

### Fourth Harmonic Secular Quantities

$$l_{42}=-\frac{45}{128}\left(\frac{r_e}{p}\right)^4n(1-e^2)^{1/2}(3-30\cos^2 I+35\cos^4 I)t$$

$$g_{42}=-\frac{15}{128}\left(\frac{r_e}{p}\right)^4n[3(4+3e^2)-18(8+7e^2)\cos^2 I+7(28+27e^2)\cos^4 I]t$$

$$h_{42}=-\frac{15}{32}\left(\frac{r_e}{p}\right)^4n(2+3e^2)\cos I(3-7\cos^2 I)t.$$

With $\sigma_4 \equiv J_4 + J_2^2$, next compute the variations in the Delaunay variables

$$\delta L = J_3 L_3 + \sigma_4 L_4 \qquad \delta H = 0$$

$$\delta G = J_3 G_3 + \frac{J_3}{J_2}\tilde{G}_3 + \sigma_4 G_4 + \frac{\sigma_4}{J_2}\tilde{G}_4$$

$$\delta l = J_3 l_3 + \frac{J_3}{J_2}\tilde{l}_3 + \sigma_4(l_{41} + l_{42}) + \frac{\sigma_4}{J_2}\tilde{l}_4$$

$$\delta g = J_3 g_3 + \frac{J_3}{J_2}\tilde{g}_3 + \sigma_4(g_{41} + g_{42}) + \frac{\sigma_4}{J_2}\tilde{g}_4$$

$$\delta h = J_3 h_3 + \frac{J_3}{J_2}\tilde{h}_3 + \sigma_4(h_{41} + h_{42}) + \frac{\sigma_4}{J_2}\tilde{h}_4$$

The variations of the elements $a$, $e$, and $\eta_0$ are then

$$\delta a = \frac{2}{an}\delta L \qquad \delta e = \frac{pn}{\mu e}\delta L - (ae)^{-1}\left(\frac{p}{\mu}\right)^{1/2}\delta G \qquad \delta\eta_0 = \frac{1-\eta_0^2}{\mu\eta_0 p}\delta G$$

The variations in the uniformising variables $E$, $v$, and $\psi$ are

$$\delta E = (\rho/a)(\delta l + \sin E\delta e)$$

$$\delta v = (1-e^2)^{1/2}\,(\rho/a)[\delta E + (1-e^2)^{-1}\sin E\delta e]$$

$$\delta\psi = \delta v + \delta g$$

The variations in the spheroidal coordinates $\rho$, $\eta$, and $\phi$ are then

$$\delta\rho = (\rho/a)\delta a - a\cos E\delta e + ae\sin E\delta E$$

$$\delta\eta = (\eta/\eta_0)\delta\eta_0 + \eta_0\cos\psi\delta\psi$$

$$\delta\phi = \delta h + (1-\eta_0^2\sin^2\psi)^{-1}\cos I[\delta\psi - \frac{1}{2}(\mu p)^{-1/2}\sin 2\psi\delta G]$$

The final rectangular coordinates $X+\delta X$, $Y+\delta Y$, and $Z+\delta Z$ are then given by

$$X + \delta X + i(Y+\delta Y) = [(\rho+\delta\rho)^2 + c^2]^{1/2}\,[1-(\eta+\delta\eta)^2]^{1/2}\exp i(\phi+\delta\phi)$$

$$\delta Z = \rho\delta\eta + \eta\delta\rho.$$

## 12. Discussion of Results

Since the chosen accuracy of the quantities appearing in the present perturbation, with factor of order $J_2^2$, is only that of an elliptic approximation, the variations in the Delaunay variables may be checked against those found by Brouwer [1959], by Garfinkel [1959], or by Kozai [1962]. Comparison of the above results with those of Brouwer or of Garfinkel shows that the long-periodic effects of the third and fourth harmonics and the secular effects of the fourth agree with Brouwer's or Garfinkel's, provided that one replaces their $J_4$ by $J_4 + J_2^2$. Comparison with Kozai shows that the short-periodic effects agree with his.[1] Similarly one can read out of Kozai's paper the long-periodic effects of $J_5$, $J_7$, and $J_9$ and the long-periodic and secular effects of $J_6$ and $J_8$; in so doing one ought in principle to replace his $J_6$ by $J_6 - J_2^3$ and his $J_8$ by $J_8 + J_2^4$, but this would be going beyond the accuracy of the present calculation. Since the author's orbital elements differ from Kozai's by terms of order $J_2$, the agreement with Kozai holds only through terms of order $J_2$ for long-periodic effects and through terms of order $J_2^2$ for short-periodic effects.

[1] On his page 451, however, in the first line for $\Delta G$, the expression cos $2g$ should be cos $g$.

To compare accuracies, we construct the following table, noting that the author's reference orbit accounts for about 99.5 percent of the deviation of the earth's potential from spherical symmetry. Thus my perturbation potential is only about 0.5 percent of Kozai's.

| | Effects of 99.5 percent of deviation from sphericity | | |
|---|---|---|---|
| | Secular accuracy | Short-periodic accuracy | Long-periodic accuracy |
| Kozai | Through $J_2^2$ | Through $J_2^2$ | Terms do not exist |
| Author | Exact | Through $J_2^2$ | Terms do not exist |

| | Effects of remaining 0.5 percent of deviation from spherity | | |
|---|---|---|---|
| | Secular accuracy | Short-periodic accuracy | Long-periodic accuracy |
| Kozai | Through $J_2^2$ | Through $J_2^2$ | Through $J_2^2$ |
| Author | Through $J_2^2$ | Through $J_2^2$ | Through $J_2$ |

Thus the advantages of the author's treatment are the exact solution for the secular effects arising from 99.5 percent of the aspherical deviation and the much shorter algorithm. The principal advantage of Kozai's treatment, arising in connection with the remaining 0.5 percent of the aspherical deviation, is his more accurate solution for the long-periodic terms.

The present solution, like all previous perturbation theories, gives rise to the resonance denominator $1-5\cos^2 I$ in some of the long-periodic terms. These terms are thus not reliable if one considers inclinations $I$ sufficiently close to 63.4° or 116.6°. For such inclinations one could improve the accuracy by boldly dropping the long-periodic terms with coefficient $J_4+J_2^2$ or, better, by superposing on the present treatment Izsak's [1962] solution of the problem of the critical inclination.

The element $e$ occurs in the denominators of $\delta e$, $l_3$, $g_3$, $\tilde{l}_3$, $\tilde{g}_3$, $l_{41}$, and $g_{41}$ and thus also in the denominators of $\delta l$, $\delta g$, $\delta E$, $\delta v$, and $\delta\psi$. No corresponding trouble occurs in the coordinates, however. To test this point, reject all terms except those containing $e^{-1}$. One then finds $l_3=-g_3$, $\tilde{l}_3=-\tilde{g}_3$, $l_{41}=-g_{41}$, $L_3=G_3$, $L_4=G_4$, and $\tilde{G}_3=\tilde{G}_4=0$. Then $\delta e=O(e^0)$, $\delta E=\delta v=\delta l$, and $\delta\psi=\delta v+\delta g=\delta l+\delta g=0$. Trouble could occur in $\delta\eta$ or in $\delta\phi$ only through $\delta\psi$, so that $\delta\eta$ and $\delta\phi$ do not become infinite for $e=0$. Similarly, trouble could occur in $\delta\rho$ through the term $-a\cos E\delta e$, which, however, does not become infinite, or through the term $ae\sin E\delta E$. But $\delta E=\delta l=O(e^{-1})$, so that this term also remains finite.

The quantity $\eta_0\equiv\sin I$ occurs in the denominators of $\delta\eta_0$, $g_3$, $h_3$, $\tilde{g}_3$, and $\tilde{h}_3$, and thus also in the denominators of $\delta g$, $\delta h$, $\delta\psi$, and $\delta\phi$. Again, however, the coordinates remain finite when $\sin I=0$. To test this point, reject all terms except those containing $\eta_0^{-1}=\csc I$. Since $G_3$ and $\tilde{G}_3$ are of order $\eta_0$ and $G_4$ and $\tilde{G}_4$ of order $\eta_0^2$, it follows that $\delta G$ is also of order $\eta_0$ and $\delta\eta_0=(1-\eta_0^2)\delta G/(\eta_0\mu p)$ remains finite. Thus the term $\sin\psi\delta\eta_0$ in $\delta\eta$ remains finite. We also have $g_3=-h_3=O(\eta_0^{-1})$, $\tilde{g}_3=\frac{1}{2}er_ep^{-1}\times\cos^2 I\csc I\cos g$, $\tilde{h}_3=-\frac{1}{2}er_ep^{-1}\cos I\csc I\cos g$, and $\delta\psi=\delta g=O(\eta_0^{-1})$. Thus the term $\eta_0\cos\psi\delta\psi$ in $\delta\eta$ also remains finite, so that $\delta\eta$ remains finite. Finally, $\delta\phi=\delta h+\cos I\delta g=J_3(g_3+h_3)+J_3J_2^{-1}(\tilde{h}_3+\tilde{g}_3\cos I)=\frac{1}{2}J_3J_2^{-1}er_ep^{-1}\cos g\csc I(\cos^3 I-\cos I)=-\frac{1}{2}J_3J_2^{-1}er_ep^{-1}\sin I\cos I\cos g=0$. Thus $\delta\phi$ remains finite. There is no trouble with $\delta\rho$.

## 13. References


Brouwer, D. 1959, Astron J. **64,** 378–397.

Garfinkel, B. 1959, Astron. J. **64,** 353–367.

Garfinkel, B. 1960, Notes of the Summer Institute in Dynamical Astronomy at Yale University, July 1960, page 281.

Izsak, I. 1962, Smithsonian Astrophysical Observatory, Special Report No. 60.

Kaula, W. 1962, Advances in Geophysics, Vol. IX, edited by H. E. Landsberg and J. Van Mieghem, p. 225 (Academic Press).

King-Hele, D. G., C. E. Cook, and J. M. Rees, 1963, Nature **197,** 785.

Langer, R. M. and J. P. Vinti, (1963) J. Res. NBS **67C** (Eng. and Instr.) No. 3, 247–249.

Vinti, J. P. 1959a, Phys. Rev. Letters **3,** 8; 1959b, J. Res. NBS **63B** (Math. and Math. Phys.) No. 2, 105–116; 1961a, J. Res. NBS **65B** (Math. and Math. Phys.) No. 3, 169–201; 1961*b*, Astron. J. **66,** 514–516; 1962, J. Res. NBS, **66B** (Math. and Math. Phys.) No. 1, 5–13.



(Paper 67B4–103)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 4, October–December 1963

# Optimal Periodic Inspection Programs for Randomly Failing Equipment

George H. Weiss*

(July 16, 1963)

There have been many analyses made of models for equipment inspection, i.e., where a system may suffer a breakdown, but such an event is only discovered by an inspection. Most analyses assume that the time to failure follows a negative exponential law which implies that only periodic inspection programs need be considered. Another model which has been analyzed by Barlow, Hunter, and Proschan finds the optimal program of inspections when the equipment reliability function is of general form, but a particular loss function is given. In this paper we find the optimal periodic inspection program for systems which do not have negative exponential reliability functions. These programs have the virtue of simplicity even though they may not be optimal in an absolute sense. Besides the periodic inspection programs, we derive results for random inspection programs.

## 1. Introduction

There have been many papers published on the subject of determining optimum checkout intervals for systems with random failure characteristics, [1–5].[1] The objective of any checkout program is to minimize the down time due to system failure. When the reliability function which describes the system's failure characteristics is negative exponential the optimal checkout policy must be a periodic one. If the system reliability function differs from the exponential, one does not expect such a simple result to obtain. Indeed Barlow, Proschan, and Hunter [3, 4] have devised an algorithm for calculating the optimum checkout schedule for a system with a general reliability function. They minimized the total loss incurred by inspection costs and the cost of uninspected down time to derive an optimum policy. This policy is, however, a difficult one to actually compute since all of the inspection times are a function of the first inspection time. This parameter must be varied in order to find the optimal policy. It has been found that the optimal inspection periods are extremely sensitive to the choice of the first period, hence a good deal of accuracy is needed in the calculations. Other studies of the problem have used the assumption of an exponential reliability function and found the optimal periodic solution [1–5].

It is the purpose of this paper to find the optimal periodic solution for the checkout interval of a system which does not necessarily have an exponential reliability function. Clearly, such a policy will not be optimal in the sense of Barlow, Proschan, and Hunter's results. However the periodic inspection program does have the advantage of simplicity and it is the sort that would most likely be followed in practice. The exact class of policies which will concern us will be denoted by $(\Delta, T)$ policies. The parameter $\Delta$ is the inspection interval and $T$ is a time at which the system is to be replaced by a new one if it has passed all of the previous inspections. Three possibilities may be considered: $\Delta$ can be held fixed and $T$ varied; $T$ can be held fixed and $\Delta$ varied; or both parameters can be varied. In the case of an exponential reliability function or no wearout, $(\Delta, \infty)$ policies would be the only logical ones to consider if inspection is considered to be perfect. We may wish to consider this possibility for the general reliability function as well. In addition to an analysis of periodic inspection policies we shall also consider random inspection policies in which the periods between successive inspections are identically distributed random variables.

## 2. The $(\Delta, \infty)$ Policy

Our criterion for determining the best $(\Delta, T)$ policy will be the parameter set which maximizes the expected operational readiness of the system. Thus, implicitly the cost function will be measured in lost time; time that is spent in inspection and replacement. The present mathematical model is therefore applicable to computer systems and to other systems which are operated continually.

We assume that system failure is complete, and that the system is replaced rather than repaired when breakdown occurs. The system failure characteristics will be described by a reliability function $R(t)$ (the probability that a system installed at $t=0$ will survive for a time $t$ or greater). It will be assumed that the first moment

$$\mu = \int_0^\infty R(t)dt \tag{1}$$

is finite. If the inspection and replacement times are assumed to be random variables then one can dis-

*Present Address: Rockefeller Institute, New York, N.Y.

[1] Figures in brackets indicate the literature references at the end of this paper.

699-888 O-63-3

tinguish between two types of models. In the first, inspections are scheduled for times $\Delta, 2\Delta, 3\Delta, \ldots$ regardless of the amount of time spent in the last inspection. In the second model the beginning of the next inspection is scheduled $\Delta$ units of time after the end of the present one, i.e., inspection is scheduled after a usage period of $\Delta$. If the inspection and replacement times are fixed rather than random the two models are formally identical after a redefinition of $\Delta$. In this paper we will treat the case of a usage period of $\Delta$. Finally, we will assume that the system is turned off during inspection, i.e., that it does not age during an inspection.

The notation in addition to that already defined is:

$\tau_i$ = time for a single inspection.
$\tau_r$ = time for a single replacement.
$\theta$ = probability of not detecting on inspection a system that has failed.

The first case to be studied will be the $(\Delta, \infty)$ policy, that is, inspection is continued until a system failure occurs. Let us consider the history of a single system and enumerate the times which will be necessary for our analysis. The nonoperational times will be of three types: An installment time; $n+1$ inspection periods, where $n$ is a random variable; and a period when the system is inoperative but that condition is not yet determined by inspection. There will also be $n+1$ periods of operation, at least $n$ of them being of duration $\Delta$, the remaining one being of duration less than or equal to $\Delta$. For a single system the time history is summarized in figure 1. Let $T$ be the lifetime of a single system measured from its installation to the installation of the next system. Then $T$ is the sum of two parts:

$$T = T_0 + T_N \tag{2}$$

where $T_0$ is the operational time and $T_N$ is nonoperational time. Barlow and Hunter have shown in [6] that the operational readiness, or proportion of time that the system is operational, is given in the limit of $t \to \infty$ by

$$P = \frac{E(T_0)}{E(T_0) + E(T_N)}. \tag{3}$$

where $E(T)$ is the expected value of $T$. This result is also a simple consequence of renewal theoretic considerations, [7]. Hence in order to evaluate $P$ we must only calculate $E(T_0)$ and $E(T_N)$ for a single system.

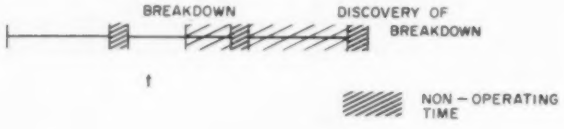


FIGURE 1. *Representation of the history of a piece of equipment.*

The calculation of $E(T_0)$ is a simple one, since it is just the operational time till failure. This is just

$$E(T_0) = \mu \tag{4}$$

where $\mu$ has been assumed finite. The calculation of $E(T_N)$ is slightly more complicated. $T_N$ can be written

$$T_N = T_r + T_1 + T_2 + \ldots + T_{n+1} + \zeta \tag{5}$$

where $T_r$ is the replacement, or installation time, $T_1$, $T_2, \ldots, T_{n+1}$ are the inspection times till just after the failure ($n$ is a random variable) and $\zeta$ is the time lost due to faulty inspection. If the expectation of this equation is taken, we find

$$E(T_N) = \tau_r + \tau_i E(n+1) + E(\zeta) \tag{6}$$

where the $\tau$'s have already been defined. The random variable $n$ has a distribution

$$Pr\{\text{exactly } n \text{ operational periods}\} = R(n\Delta) - R((n+1)\Delta). \tag{7}$$

Hence $E(n+1)$ is

$$E(n+1) = \sum_{n=0}^{\infty} (n+1)\,[R(n\Delta) - R((n+1)\Delta)] \tag{8}$$
$$= \sum_{n=0}^{\infty} R(n\Delta) = G(\Delta)$$

where the finiteness of the first moment insures the convergence of this series. Finally, the residual nonoperational time is composed of the random time between the failure and the first inspection thereafter and a random period due to the effect of imperfect inspection. The probability density of a system failure is $-R'(t)$, which implies that $E(\zeta)$ is given by

$$\begin{aligned} E(\zeta) &= -\sum_{n=0}^{\infty} \int_{n\Delta}^{(n+1)\Delta} ((n+1)\Delta - t) R'(t)\,dt \\ &\quad + (1-\theta) \sum_{m=0}^{\infty} m\theta^m (\tau_i + \Delta) \\ &= \sum_{n=0}^{\infty} \Delta R(n\Delta) - \mu + \frac{\theta}{1-\theta}(\tau_i + \Delta) \\ &= \Delta G(\Delta) - \mu + \frac{\theta}{1-\theta}(\tau_i + \Delta). \end{aligned} \tag{9}$$

Combining eqs (3), (6), (8), and (9), we find for the operational readiness

$$\begin{aligned} P(\Delta) &= \frac{\mu}{\tau_r + (\tau_i + \Delta)\left(G(\Delta) + \dfrac{\theta}{1-\theta}\right)} \\ &= \frac{1}{\lambda_r + \left(\lambda_i + \dfrac{\Delta}{\mu}\right)\left(G(\Delta) + \dfrac{\theta}{1-\theta}\right)} \end{aligned} \tag{10}$$

where we have introduced the dimensionless parameters

$$\lambda_r = \tau_r/\mu,\ \lambda_i = \tau_i/\mu. \tag{11}$$

Occasionally useful bounds on $P(\Delta)$ can be obtained by use of the inequalities

$$\Delta + \mu > \Delta G(\Delta) > \mu \tag{12}$$

This is easily derived by reference to figure 2. There it is seen that $\Delta G(\Delta)$ is the area under the upper set of rectangles, which is greater than the area under the curve for $R(t)$. Furthermore the area under the curve for $R(t)$ is greater than that under the lower set of rectangles. This latter area is, however $\Delta \sum_{n=1}^{\infty} R(n\Delta) = \Delta(G(\Delta)-1)$ from which the upper bound follows. The inequalities of eq (12) thus yield bounds on $P(\Delta)$ as follows

$$\frac{1}{\lambda_r + \left(\lambda_i + \frac{\Delta}{\mu}\right)\left(\frac{\mu}{\Delta} + \frac{1}{1-\theta}\right)} < P(\Delta) < \frac{1}{\lambda_r + \left(\lambda_i + \frac{\Delta}{\mu}\right)\left(\frac{\mu}{\Delta} + \frac{\theta}{1-\theta}\right)}. \tag{13}$$

If we write this inequality as $L(\Delta) < P(\Delta) < U(\Delta)$ then it is interesting to note that both $L(\Delta)$ and $U(\Delta)$ have maxima. The maximum of $L(\Delta)$ occurs at

$$\Delta_m = \mu\sqrt{\lambda_i(1-\theta)} \tag{14}$$

and the maximum of $U(\Delta)$ occurs at

$$\Delta_m' = \frac{\Delta_m}{\sqrt{\theta}} \tag{15}$$

Thus the maximum value of $P(\Delta)$ obeys the inequalities

$$\frac{1}{\lambda_r + \left(1 + \sqrt{\frac{\lambda_i}{1-\theta}}\right)^2} < P_{max} < \frac{1}{\lambda_r + \left(1 + \sqrt{\frac{\lambda_i\theta}{1-\theta}}\right)^2}. \tag{16}$$

The optimal value of $\Delta$, call it $\Delta_0$, is the solution to

$$G(\Delta_0) + \frac{\theta}{1-\theta} + (\tau_i + \Delta_0)G'(\Delta_0) = 0. \tag{17}$$

and the value of $P(\Delta_0)$ is

$$P(\Delta_0) = \frac{\mu}{\tau_r - (\tau_i + \Delta_0)^2 G'(\Delta_0)}. \tag{18}$$

When $\tau_i/\mu$ is small in comparison to 1 we can derive an approximate expression for $\Delta_0$. If $\tau_i$ were identically equal to zero, that is, if inspection takes no time to carry out, then $\Delta_0 = 0$, or continuous monitoring is the optimal solution. If $\tau_i/\theta$ is small but not identically equal to zero, it may be supposed that $\Delta_0/\mu$ will also be a small quantity. Let us then expand $G(\Delta_0)$ and $G'(\Delta_0)$ for small $\Delta_0$ by means of the Euler-Maclauren expansion, [8]

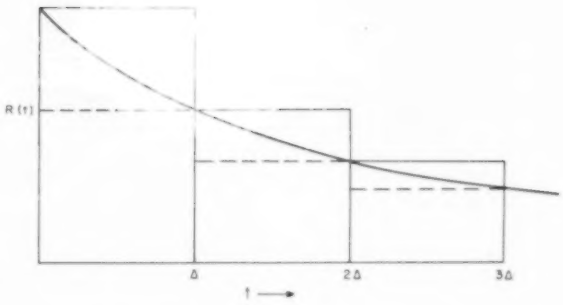


FIGURE 2. *Plot of* R(t) *and upper and lower bounding functions.*

$$G(\Delta) = \sum_{n=0}^{\infty} R(n\Delta) = \frac{\mu}{\Delta} + \frac{1}{2} - \frac{R'(0)}{12}\Delta + 0(\Delta^2)$$
$$G'(\Delta) = -\frac{\mu}{\Delta^2} - \frac{R'(0)}{12} + 0(\Delta). \tag{19}$$

In this approximation the equation for the optimal $\Delta_0$ becomes

$$\frac{\mu}{\Delta_0} + \frac{1}{2} + \frac{\theta}{1-\theta} - \mu\left(\frac{\tau_i}{\Delta_0^2} + \frac{1}{\Delta_0}\right) = 0. \tag{20}$$

We have neglected terms $R'(0)\Delta$ and $R'(0)\tau_i$ which are presumed small in comparison to terms like 1/2 and $\tau_i\mu/\Delta^2$ in the present calculation. Equation (20) has the solution

$$\frac{\Delta_0}{\mu} = \sqrt{\frac{2(1-\theta)}{1+\theta}\frac{\tau_i}{\mu}} + 0\left(\frac{\tau_i}{\mu}\right) \tag{21}$$

which generalizes a result found earlier for the exponential reliability function [5]. To this approximation the value of $P_{max}$ is

$$P_{max} = \frac{2(1-\theta)}{2\lambda_r(1-\theta) + (\sqrt{\lambda_i(1+\theta)} + \sqrt{2(1-\theta)})^2} \tag{22}$$

In figure 3 we have plotted $P(\Delta/T)$ for the reliability function $R(t) = \exp(-t^2/T^2)$ for $\Delta/T$ in the range 0 to 0.20 and $\theta = 0$ and $\theta = 0.25$. It can be observed that the variation of the maximum with $\Delta/T$ is negligible in the range that we have considered.

## 3. Random Inspection Periods

Thus far we have considered an idealized case in which inspections are performed as scheduled. It is

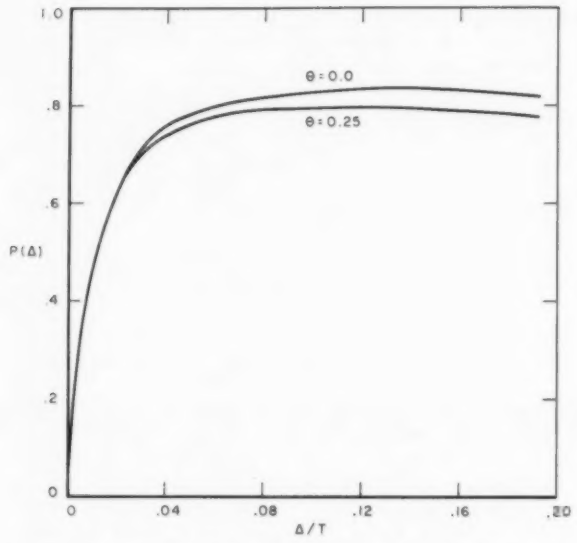


FIGURE 3. *Plot of* $P(\Delta)$ *for* $R(t)=exp\ (-t^2/T^2)$ *for* $\theta=0.25$ *and* $\lambda_i=0.01$, $\lambda_r=0.05$.

of some interest to consider the possibility of random examinations, since this, rather than perfect control over inspection is likely to be a more accurate description of actual monitoring situations. We will assume that the inspection following a given one takes place at some time $t$ after the conclusion of the first inspection. The time $t$ is a random variable which will be described by a probability density function $\Psi(t)$ with a finite first moment $\nu$.

Again, in order to calculate operational readiness we may use eq (3) with $E(T_0)=\mu$. The expression for the nonoperating time given in eq (6) is still valid but it requires somewhat more effort to evaluate $E(T_N)$. As before we need to evaluate $E(n+1)$ and $E(\zeta)$. We consider $E(n+1)$ first. For the evaluation of this quantity we define a probability density $w_n(t)$ by

$$w_n(t)dt = Pr\{\text{sum of } n \text{ periods of operating time is in } (t, t+dt)\}. \tag{23}$$

The probability that there are exactly $n$ complete operating periods without a system failure is

$$\Omega_n = \int_0^\infty w_n(t)R(t)dt - \int_0^\infty w_{n+1}(t)R(t)dt \tag{24}$$

since the first integral is the probability that the system is reliable through at least $n$ inspections and the second integral has a similar interpretation. The expectation of $n+1$ is therefore

$$\begin{aligned} E(n+1) &= \sum_{n=0}^{\infty}(n+1)\left\{\int_0^\infty R(t)\,[w_n(t)-w_{n+1}(t)]dt\right\} \\ &= 1+\int_0^\infty R(t)w(t)dt \end{aligned} \tag{25}$$

where

$$w(t)=\sum_{n=1}^{\infty} w_n(t) \tag{26}$$

and we have used as a definition

$$w_0(t)=\delta(t). \tag{27}$$

Since the $w_n(t)$ satisfy the recurrence relation

$$\begin{aligned} w_{n+1}(t) &= \int_0^t w_n(\tau)\Psi(t-\tau)d\tau \\ w_1(t) &= \Psi(t), \end{aligned} \tag{28}$$

it is easily verified, by summing both sides of this equation, that $w(t)$ is a solution of the renewal equation

$$w(t)=\Psi(t)+\int_0^t w(\tau)\Psi(t-\tau)d\tau. \tag{29}$$

Notice that $w(t)dt$ is the probability that some inspection occurs after an amount of operating time between $t$ and $t+dt$.

Finally, we must evaluate $E(\zeta)$. Let us assume that there is a system failure after $\tau$ hours of operating time. The first component of $\zeta$ is the time to the immediately following inspection period. A second component will have to be added in case $\theta \neq 0$. Let the time from the system failure to the immediately following inspection be $t$ (in the terminology of renewal theory this is the forward delay time). In order to calculate the distribution of the forward delay we must take two possibilities into account; either the next inspection is the first one, or else the last inspection ended in the time interval $(x, x+dx)$ with probability $w(x)dx$. These last two events combine to give a result for the forward delay density at time $\tau$, $\rho(\tau, t)$,

$$\rho(\tau, t)=\Psi(\tau+t)+\int_0^\tau w(x)\Psi(\tau+t-x)dx. \tag{30}$$

The first component of $E(\zeta)$ is

$$E_1(\zeta)=-\int_0^\infty d\tau\int_0^\infty dt\ t\rho(\tau, t)R'(\tau) \tag{31}$$

where we have averaged over all possible failure times. The second component of $E(\zeta)$ is the expected down time due to undiscovered failure. The probability that it will take exactly $n$ inspections to discover the failure is $(1-\theta)\theta^{n-1}$ and the expected amount of down time consumed before the discovery is made is

$$E_2(\zeta)=(1-\theta)\sum_{n=1}^{\infty}\theta^{n-1}(n-1)\,(\nu+\tau_i)=\frac{\theta(\nu+\tau_i)}{1-\theta} \tag{32}$$

since $\nu$ is the expected time between two consecutive inspections. Combining the expressions for $E_1(\zeta)$, $E_2(\zeta)$ and $E(n+1)$ we have the final result

$$P=\frac{\mu}{\tau_r+\mu+\tau_i\left[1+\int_0^\infty R(t)w(t)dt\right]+\dfrac{\theta(\nu+\tau_i)}{1-\theta}-\int_0^\infty d\tau\int_0^\infty dt\,t\rho(\tau,t)R'(\tau)}. \tag{33}$$

In the particular case of a completely random examination policy,

$$\Psi(t)=\sigma e^{-\sigma t} \tag{34}$$

all of the quantities required by eq (33) are readily evaluated. It is well known and easily verified that for the negative exponential distribution one has

$$\begin{gathered} w(t)=\sigma,\ \nu=1/\sigma \\ \rho(\tau,t)=\sigma e^{-\sigma t}. \end{gathered} \tag{35}$$

Thus we find that

$$\begin{gathered} -\int_0^\infty R'(\tau)d\tau\int_0^\infty \sigma t e^{-\sigma t}dt=\frac{1}{\sigma}R(\tau)\Big|_0^\infty=\frac{1}{\sigma}, \\ \int_0^\infty R(t)w(t)dt=\sigma\int_0^\infty R(t)dt=\sigma\mu. \end{gathered} \tag{36}$$

The value of $P$ can be given explicitly as

$$P=\frac{\sigma}{\sigma(\lambda_r+1)+\lambda_i\sigma\left(\dfrac{1}{1-\theta}+\sigma\mu\right)+\dfrac{1}{(1-\theta)\mu}} \tag{37}$$

where the dimensionless parameters $\lambda_i$ and $\lambda_r$ have been defined before. The solution for the maximizing value of $\sigma$ is

$$\sigma_0=\frac{1}{\mu\sqrt{\lambda_i(1-\theta)}} \tag{38}$$

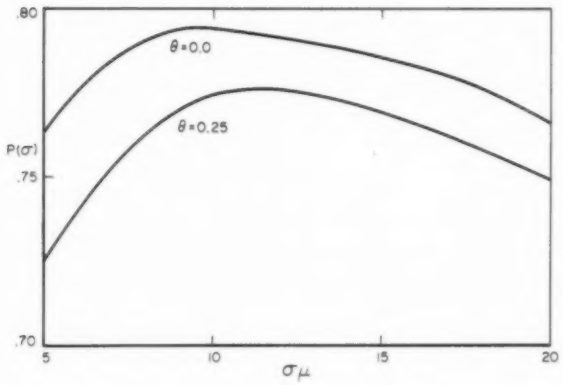


FIGURE 4. *Plot of* P (σ) *for random maintenance for* $\theta=0.0, 0.25$ *and* $\lambda_i=0.01, \lambda_r=0.05$.

and the maximum value of $P$ is

$$P_{max}=\left(1+\lambda_r+\frac{\lambda_i}{1-\theta}+2\sqrt{\frac{\lambda_i}{1-\theta}}\right)^{-1} \tag{39}$$

It is interesting to notice that this result does not depend on $R(t)$ except through $\mu$.

In figure 4 we have plotted $P$ as a function of $\sigma\mu$ for $\lambda_r=0.05$, $\lambda_i=0.01$, and $\theta=0$ and 0.25. It will be observed that the curve is rather flat around the maximum, and the maximum itself does not shift appreciably as $\theta$ is changed.

## 4. The (Δ, NΔ) Case

When the reliability function drops off much more quickly than a negative exponential, it is plausible that one might want to replace the equipment after a specified time rather than continuing to inspect it. Even if the reliability function is a negative exponential, finite scheduled replacement times would be indicated if the quality of inspection is poor, i.e., if $\theta$ is close to 1. Quite generally, one might consider a $(\Delta, N\Delta)$ policy in which replacement takes place after $N\Delta$ units of operating time. In this way the number $N$ becomes an adjustable parameter in the problem, and the policy can be maximized over both $\Delta$ and $N$. This procedure leads to quite complicated expressions. Let us therefore restrict our efforts to a comparison of $(\Delta, \Delta)$ and $(\Delta, \infty)$ policies for particular parameter values.

For the $(\Delta, \Delta)$ policy we note that

$$E(T_0)=\int_0^\Delta R(x)dx=\mu(\Delta) \tag{40}$$

and

$$E(T_N)=\tau_r+\Delta-E(T_0). \tag{41}$$

These results are to be substituted into eq (3) to obtain the expression for $P(\Delta)$

$$P(\Delta)=\frac{\int_0^\Delta R(x)dx}{\Delta+\tau_r}. \tag{42}$$

In figure 5 we have plotted $P(\Delta)$ as a function of $\Delta/T$ for $R(t)=\exp(-t/T)$, for the parameter values $\lambda_r=\tau_r/T=0.05$ and $\lambda_i=\tau_i/T=0.01$. When $\theta=0$ one can do better with a $(\Delta, \infty)$ policy than with a $(\Delta, \Delta)$ policy although when $\theta=0.9$ this is no longer the case. For other parameter values there may exist an $N$ such that the $(\Delta, N\Delta)$ policy is better than either the $(\Delta, \Delta)$ policy or the $(\Delta, \infty)$ policy. The value of $P(\Delta)$ for the $(\Delta, N\Delta)$ policy can be calculated from $E(T_N)$ given by

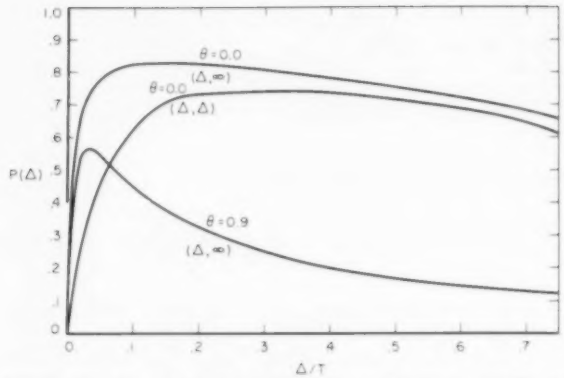


FIGURE 5. *Plot of* $P(\Delta)$ *against* $\Delta/T$ *for* $R(x)=exp\ (-x/T)$ *for* $\lambda_i=0.01$, $\lambda_r=0.05$.

$$E(T_N)=\tau_r-(1-\theta)\sum_{j=0}^{N-2}\sum_{k=1}^{N-j-1}\theta^{k-1}\int_{j\Delta}^{(j+1)\Delta}[j\Delta+k(\Delta+\tau_i)-x]R'(x)dx$$
$$-\sum_{j=0}^{N-2}\theta^{N-j-1}\int_{j\Delta}^{(j+1)\Delta}[N\Delta+(N-j-1)\tau_i-x]R'(x)dx$$
$$-\int_{(N-1)\Delta}^{N\Delta}(N\Delta-x)R'(x)dx \qquad (43)$$

and

$$E(T_0)=\int_0^{N\Delta}R(x)dx. \qquad (44)$$

The expression of eq (43) is derived by noting that if the system fails at time $x$ in the interval $[j\Delta, (j+1)\Delta]$ then the time loss due to inspections and to down time is $j\Delta+k(\Delta+\tau_i)-x$ if $k-1$ inspections have been incorrect, accounting for the first set of terms. If all inspections are faulty then the term $1-\theta$ is to be omitted and the second set of terms results. Finally, if the system fails in $[(N-1)\Delta, N\Delta]$ there are no inspections so that the last term has no $\theta$ dependence. The expression for $P(\Delta)$ which results from the use of eqs (43) and (44) is so cumbersome that no information can be extracted from it without considerable computation.

---

I thank the readers Joan Rosenblatt and Joel Levy for detecting several errors in the original manuscript and for suggesting many improvements.

## 4. References

[1] M. Kamins, Determining checkout intervals for systems subject to random failures, Rand Research Memo RM-2578.
[2] J. Coleman, I. Abrams, Mathematical model for operation readiness, J. Operations Res. **10**, 126 (1962).
[3] R. E. Barlow, L. Hunter, F. Proschan, Optimum monitoring procedures, Boeing scientific Res. Labs. Document, DI-82-0104, 1961.
[4] L. C. Hunter, Optimum checking intervals, in Statistical Theory of Reliability (University of Wisconsin Press) 1963.
[5] G. H. Weiss, A problem in equipment maintenance, Management Science **8**, 266 (1962).
[6] R. E. Barlow, L. C. Hunter, Optimum preventive maintenance policies, J. Operations Res. **8**, 90 (1960).
[7] D. R. Cox, Renewal theory (Methuen, London) 1962.
[8] E. T. Whittaker, G. N. Watson, A course of modern analysis, (Cambridge Press) Fourth edition, 1952.

(Paper 67B4-104)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 4, October–December 1963

# An Analysis of Pedestrian Queueing

George H. Weiss*

(June 10, 1963)

This paper considers the steady state queueing properties of pedestrians who are delayed by traffic on a main road. It is assumed that the size of the gap between two successive cars determines the probability that an individual pedestrian or a group of pedestrians will cross. The technique of imbedded Markov chains yields many new results as well as some previously found, by combinatorial methods by Tanner in 1951.

## 1. Introduction

There has been a recent increase of interest in the problem of the delay to traffic at a stop sign. This problem was first suggested by data analyzed by Adams in 1936, [1].[1] Adams derived an expression for the mean delay to a pedestrian caused by a stream of traffic on a main highway, in which the successive gaps (in time) are independent random variables, each with a negative exponential distribution. It was assumed that each pedestrian waits for a gap which is larger than a prescribed safe interval before attempting to cross. Garwood, in 1941 [2], took up the problem in a modified form. He dealt with the problem of driver delay at an intersection in which a traffic signal is partially operated by the main stream traffic and partially by the crossing traffic. When a crossing car arrives at the intersection it actuates the signal so that the light will change to green either at a fixed time $M$ after its arrival, or at some earlier time if the main stream is sufficiently light. Garwood's analysis assumed a negative exponential distribution for headways. Garwood derived an exact distribution for the delay. The next discussion of the delay problem was given by Raff, [3], who also assumed a negative exponential gap distribution and a gap acceptance function given by a step function, i.e., cross if the gap is greater than a given $T_c$. The same problem was taken up in much greater detail at about the same time by Tanner [4] who derived a large number of results for the statistics of the size of the queue of pedestrians. He assumed that the pedestrians' arrival distribution was Poisson, as well as that of the traffic, and calculated such quantities as the expected number of pedestrians crossing between successive cars, the size of the queue at a random time, and others which will be reproduced in the present paper. The basic assumption made by Tanner is that of a step function distribution for the gap acceptance function both for an individual pedestrian and for the group. We might note that the process described by Tanner, and indeed that which is described below in the following manuscript, is a bulk service process in which a stationary state is always reached for all gap distributions observed in moderate traffic. A later contribution to the pedestrian delay problem was made by Mayne, [5], who showed, by fairly involved calculations, how to calculate the delay to a single pedestrian when the gap distribution was other than Poisson, but otherwise assuming a step gap acceptance function.

All of the work mentioned above was carried out by means of combinatorial arguments, which, except in the case of a negative exponential gap distribution, were quite complicated. Recent work on delay problems has proceeded from the observation that the passage of a car on the main highway is a regeneration point, and the techniques of renewal theory can be used to solve delay problems in a way which almost completely eliminates combinatorial reasoning from the solution. As an example of papers which make use of renewal-theoretic techniques we cite [6–9]. Because the solution of delay problems has now been freed of the complications of the earlier analyses it is

*Present address: Rockefeller Institute, New York, N.Y.

[1]Figures in brackets indicate the literature references at the end of this paper.

natural to attack more realistic situations. Thus, in the first paper on renewal theory as applied to the delay problem, not only was it possible to relax the restriction of a negative exponential gap distribution (this had already been done by Mayne) but it was also possible to take into account a gap acceptance function which was not a step function. Specifically, one can define a probability $\alpha(t)$, such that an individual will cross the highway with probability $\alpha(t)$ when the gap which confronts him is of duration $t$. Actual measurements of $\alpha(t)$, both on pedestrian and vehicular traffic, indicate that $\alpha(t)$ is a continuous function of $t$ rather than a step function, [10–12]. A typical form found in measurements on the vehicular gap acceptance probability is

$$\begin{aligned} \alpha(t) &= 0 \qquad 0 \leq t \leq T_c \\ &= 1 - \exp[-\beta(t - T_c)] \qquad t \geq T_c. \end{aligned} \tag{1.1}$$

The theory which has been developed and partially checked against experiment applies to a single driver at a stop sign confronted by a single lane of homogeneous traffic. Very little is known about driver behavior when confronted by a multilane highway, although it is not difficult to extend the theory to the case where the waiting driver considers the situation in each lane independently. One can also solve for the moments of the delay distribution for a simple model of nonhomogeneous traffic in which there are $N$ types of cars and the gap distribution between successive cars depends on the types involved, [6].

While the single car, or single pedestrian delay distribution is a handy function for the traffic engineer to know, it has little bearing on the queueing properties of the system. This is a difficult problem to deal with when the queues are considered to be those of cars, who must necessarily form into a line, and may be released in groups whose size depends on the size of the gap in traffic on the main highway. In this situation, where it is assumed that the second car in line takes a non-negligible time to reach the head of the line when the first car has crossed, the traffic on the main highway may indeed be so heavy as to cause an unlimited buildup of a queue on the feeder road. In the pedestrian queue first studied by Tanner all of the pedestrians in the queue see the gap simultaneously and can cross simultaneously. It is this feature that insures that except for pathologic and obviously unrealistic gap distributions, the queue will have steady state properties.

It is the purpose of this paper to reconsider the pedestrian queueing problem, eliminating the restriction to a negative exponential gap distribution and using a gap acceptance probability which is not necessarily a step function. It will be seen that by using the techniques associated with imbedded Markov chains (first introduced into this problem area by Oliver and Bisbee, [7]), the resulting calculations become simple, because of the particular form found for the transition matrix, when it is assumed that the gap acceptance function for a group of pedestrians is the same as that for a single pedestrian. We shall calculate the probability generating functions and the first two moments for the number of pedestrians in queue at each regeneration point, the number in queue at a random time, and the number crossing after a random vehicle.

## 2. General Theory

We begin by considering the following specification of the situation to be studied. We assume:

1. Poisson arrival of pedestrians, with arrival rate $\lambda$.
2. General stationary headway distribution in which successive gaps are independent random variables described by a density $\varphi(t)$.
3. The gap acceptance function for each pedestrian is $\alpha(t)$.

The first and simplest problem for pedestrian queueing is the study of the imbedded Markov chain, in which we take as the regeneration points for the process, the times of departure of cars on the highway from a point directly in front of the group of pedestrians. We shall calculate the elements of the transition matrix $\mathbf{P} = (p_{nm})$ in several situations. The element $p_{nm}$ represents the

probability that if there are $n$ pedestrians in queue at one regeneration point, there will be $m$ pedestrians in queue at the succeeding regeneration point. To calculate $p_{nm}$ we must be slightly more precise about our definition of regeneration points. It will be assumed that if a car passes in front of the queue at time $t$, any pedestrians who choose to cross will do so at a time $t+0$.

The problem which corresponds to those treated by previous authors is that in which either a pedestrian crosses on arrival at the intersection (i.e., he is not delayed at all) or else the whole group of pedestrians in queue decides to cross in a body when a new gap presents itself. The gap acceptance function for the group is taken to be that for the individual. We shall now calculate the $p_{nm}$ for this situation.

There are two cases to be examined. When $n$ is greater than $m$ the transition $n \to m$ can take place only as $n \to 0 \to m$, that is, first the queue empties entirely and then $m$ or more pedestrians arrive of which $m$ choose to join the queue. When $m$ is greater than or equal to $n$ the transition $n \to m$ can take place in two ways. The first is $n \to 0 \to m$ as already discussed, and the second is $n \to m$. In this second possibility the pedestrians in queue choose not to cross the road as the car passes and $m-n$ pedestrians arrive who choose to join the queue.

Since we are assuming Poisson arrival of pedestrians, the arrival times can be considered to be uniformly distributed in the interval $(0, t)$ for any $t$, [14]. Thus, the probability that a pedestrian will see an initial gap of $(x, x+dx)$ units of time will be $dx/t$. Therefore, the probability of immediate crossing by an arbitrary arrival in this interval is $\alpha^*(t)$, where

$$\alpha^*(t) = \frac{1}{t}\int_0^t \alpha(x)dx, \tag{2.1}$$

$\alpha^*(t)$ is the average of $\alpha(t)$ taken with a uniform weight over the interval. The matrix element $p_{nm}$ for $n > m$ is therefore

$$\begin{aligned} p_{nm}^{(1)} &= \sum_{k=0}^{\infty}\int_0^{\infty} \alpha(t)\frac{(\lambda t)^{k+m}}{(k+m)!}e^{-\lambda t}\binom{k+m}{m}[\alpha^*(t)]^k[1-\alpha^*(t)]^m\varphi(t)dt \\ &= \int_0^{\infty}\alpha(t)e^{-\lambda t[1-\alpha^*(t)]}\frac{[\lambda t(1-\alpha^*(t))]^m}{m!}\varphi(t)dt,\ n > m. \end{aligned} \tag{2.2}$$

The derivation of this relation is as follows: The transition $n \to 0$ (the entire queue crosses with the passage of a car) takes place with probability $\alpha(t)$; $k+m$ pedestrians arrive in the gap of $t$ units, with probability $e^{-\lambda t}(\lambda t)^{k+m}/(k+m)!$; of these $m$ remain in queue, where $m$ is a random variable with a binomial distribution with probability $1-\alpha^*(t)$. Notice that the second line can be summarized by assuming a Poisson arrival rate on a reduced time scale defined by

$$T(t) = t[1-\alpha^*(t)] = \int_0^t (1-\alpha(u))du. \tag{2.3}$$

When $m$ is greater than or equal to $n$, another term is needed in addition to that given in eq (2.2). This term is needed to cover the case of the direct transition $n \to m$ and is

$$\begin{aligned} p_{nm}^{(2)} &= \sum_{k=0}^{\infty}\int_0^{\infty}[1-\alpha(t)]e^{-\lambda t}\frac{(\lambda t)^{k+m-n}}{(k+m-n)!}\binom{k+m-n}{m-n}[\alpha^*(t)]^{k+m-n}[1-\alpha^*(t)]^{m-n}\varphi(t)dt \\ &= \int_0^{\infty}[1-\alpha(t)]\frac{(\lambda T(t))^{m-n}}{(m-n)!}e^{-\lambda T(t)}\varphi(t)dt \end{aligned} \tag{2.4}$$

where $T(t)$ is defined in eq (2.3). The transition matrix $\mathbf{P}$ is therefore

$$\mathbf{P}=\begin{bmatrix} \Delta_0+\epsilon_0 & \Delta_1+\epsilon_1 & \Delta_2+\epsilon_2 & \Delta_3+\epsilon_3 & \cdots \\ \Delta_0 & \Delta_1+\epsilon_0 & \Delta_2+\epsilon_1 & \Delta_3+\epsilon_2 & \cdots \\ \Delta_0 & \Delta_1 & \Delta_2+\epsilon_0 & \Delta_3+\epsilon_1 & \cdots \\ \Delta_0 & \Delta_1 & \Delta_2 & \Delta_3+\epsilon_0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \tag{2.5}$$

where [2]

$$\begin{aligned} \Delta_k &= \int_0^\infty \alpha(t)\varphi(t)\,\frac{(\lambda T)^k}{k!}\,e^{-\lambda T}dt \\ \epsilon_k &= \int_0^\infty [1-\alpha(t)]\varphi(t)\,\frac{(\lambda T)^k}{k!}\,e^{-\lambda T}dt. \end{aligned} \tag{2.6}$$

The gross properties of this Markov chain are easily deduced. Let us denote by $\boldsymbol{\theta}(n)$ the vector of state probabilities after the passage of the $n$th car, where

$$\boldsymbol{\theta}(n)=(\theta_0(n), \theta_1(n), \ldots). \tag{2.7}$$

The vector $\boldsymbol{\theta}(n)$ satisfies

$$\boldsymbol{\theta}(n+1)=\boldsymbol{\theta}(n)\mathbf{P} \tag{2.8}$$

or

$$\boldsymbol{\theta}(n)=\boldsymbol{\theta}(0)\mathbf{P}^n. \tag{2.9}$$

Because of the simple form of $\mathbf{P}$ shown in eq (2.5), eq (2.8) is most conveniently written in component form

$$\theta_k(n+1)=\Delta_k+\sum_{j=0}^{k}\theta_j(n)\epsilon_{k-j} \tag{2.10}$$

which suggests the use of generating functions. Defining the generating functions

$$\begin{aligned} \Psi_n(s)&=\sum_{k=0}^{\infty}\theta_k(n)s^k, \qquad \epsilon(s)=\sum_{k=0}^{\infty}\epsilon_k s^k \\ \Delta(s)&=\sum_{k=0}^{\infty}\Delta_k s^k, \end{aligned} \tag{2.11}$$

we find

$$\Psi_{n+1}(s)=\Delta(s)+\epsilon(s)\Psi_n(s) \tag{2.12}$$

and by iteration that

$$\Psi_n(s)=\epsilon^n(s)\Psi_0(s)+\Delta(s)\,\frac{1-\epsilon^n(s)}{1-\epsilon(s)}. \tag{2.13}$$

It is readily verified that

$$\begin{aligned} \Delta(s)&=\int_0^\infty \alpha(t)\varphi(t)e^{-\lambda T(t)(1-s)}dt \\ \epsilon(s)&=\int_0^\infty [1-\alpha(t)]\varphi(t)e^{-\lambda T(t)(1-s)}dt. \end{aligned} \tag{2.14}$$

[2] Although the derivation of $p_{0j}=\Delta_j+\epsilon_j$ is not quite correct, since one cannot speak of a group of 0 persons choosing to cross the road, the result is nonetheless correct, since $\Delta_j+\epsilon_j$ is independent of $\alpha(t)$.

It would be difficult to solve explicitly for all of the components of $\boldsymbol{\theta}(n)$. However, it is not difficult to find an expression for the moments. Let us denote by $\mu_k(n)$ the $k$th ordinary moment, that is

$$\mu_k(n) = \sum_{m=0}^{\infty} m^k \theta_m(n). \tag{2.15}$$

Noting that

$$\epsilon(1) + \Delta(1) = 1 \tag{2.16}$$

we find for the first two moments

$$\mu_1(n) = \mu_1(\infty) + \epsilon^n(1)\,[\mu_1(0) - \mu_1(\infty)]$$

$$\mu_2(n) = \mu_1(n) + \epsilon^n(1)\,[\mu_2(0) - \mu_1(0)] + \frac{1-\epsilon^n(1)}{1-\epsilon(1)} \int_0^{\infty} (\lambda T)^2 \varphi(t)\,dt$$

$$+ 2\left[\frac{1-\epsilon^n(1)}{1-\epsilon(1)}\,\mu_1(\infty) + n\epsilon^{n-1}(1)\,(\mu_1(0) - \mu_1(\infty))\right] \int_0^{\infty} \lambda T(1-\alpha(t))\varphi(t)\,dt \tag{2.17}$$

where

$$\epsilon(1) = \int_0^{\infty} [1-\alpha(t)]\varphi(t)\,dt \tag{2.18}$$

and

$$\mu_1(\infty) = \frac{\int_0^{\infty} \lambda T(t)\varphi(t)\,dt}{1-\epsilon(1)} = \frac{\int_0^{\infty} \lambda T(t)\varphi(t)\,dt}{\int_0^{\infty} \alpha(t)\varphi(t)\,dt}. \tag{2.19}$$

The equilibrium value of the second moment, $\mu_2(\infty)$ is given by

$$\mu_2(\infty) = \mu_1(\infty) + \frac{\int_0^{\infty} (\lambda T)^2 \varphi(t)\,dt}{1-\epsilon(1)} + \frac{2}{(1-\epsilon(1))^2} \left\{\int_0^{\infty} \lambda T[1-\alpha(t)]\varphi(t)\,dt\right\} \left\{\int_0^{\infty} \lambda T \varphi(t)\,dt\right\}. \tag{2.20}$$

By these methods we can determine the distribution of the several parameters discussed by Tanner and by Mayne. The probability that there are $n$ pedestrians in queue after a random vehicle is $\theta_n(\infty)$. Hence the expected number of pedestrians crossing after a random vehicle passes is

$$\sum_{n=0}^{\infty} n\theta_n(\infty) \int_0^{\infty} \alpha(t)\varphi(t)\,dt = \mu_1(\infty) \int_0^{\infty} \alpha(t)\varphi(t)\,dt. \tag{2.21}$$

The $k$th moment of the distribution is similarly equal to $\mu_k(\infty) \int_0^{\infty} \alpha(t)\varphi(t)\,dt$.

The probability that there are no pedestrians in queue when the $n$th vehicle passes is

$$\theta_0(n) = \Psi_n(0) = \epsilon^n(0)\Psi_0(0) + \Delta(0)\,\frac{1-\epsilon^n(0)}{1-\epsilon(0)} \tag{2.22}$$

where

$$\epsilon(0) = \epsilon_0 = \int_0^{\infty} \alpha(t)\varphi(t)e^{-\lambda T(t)}\,dt$$

$$\Delta(0) = \Delta_0 = \int_0^{\infty} [1-\alpha(t)]\varphi(t)e^{-\lambda T(t)}\,dt. \tag{2.23}$$

At equilibrium ($n = \infty$) we have the result

$$\theta_0(\infty) = \frac{\Delta(0)}{1-\epsilon(0)} = \frac{\Delta_0}{1-\epsilon_0}. \tag{2.24}$$

The probability density for the amount of time that any single pedestrian is delayed will be denoted by $\Omega(t)$, and can be immediately taken from the results of reference [6], since each pedestrian can be regarded as not being influenced by the presence of any others. Let us denote the Laplace transform of any function, say $F(t)$, by the same function with argument $s$ and an asterisk, e.g.,

$$F^*(s)=\int_0^\infty e^{-st}F(t)dt. \tag{2.25}$$

Then it is shown in [6] that

$$\Omega^*(s)=\int_0^\infty \alpha(t)\varphi_0(t)dt+\frac{\zeta_0^*(s)}{1-\zeta^*(s)}\int_0^\infty \alpha(t)\varphi(t)dt \tag{2.26}$$

where

$$\begin{aligned}\zeta(t)&=\varphi(t)\,[1-\alpha(t)]\\ \zeta_0(t)&=\varphi_0(t)\,[1-\alpha(t)]\end{aligned} \tag{2.27}$$

and where $\varphi_0(t)$ is the density of the first gap given by

$$\varphi_0(t)=\int_t^\infty \varphi(x)dx\Big/\int_0^\infty x\varphi(x)dx.$$

Expressions for the first two moments of the delay density are given in [6] and are repeated here for reference:

$$\begin{aligned}\bar{t}&=\eta_{01}+\frac{\eta_{00}}{1-\eta_{10}}\eta_{11}\\ \bar{t^2}&=\eta_{02}+\frac{2\eta_{01}\eta_{11}+\eta_{00}\eta_{12}}{1-\eta_{10}}+\frac{2\eta_{00}\eta_{11}^2}{(1-\eta_{10})^2}\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}\eta_{0j}&=\int_0^\infty t^j\zeta_0(t)dt=\int_0^\infty t^j\varphi_0(t)\,[1-\alpha(t)]dt\\ \eta_{1j}&=\int_0^\infty t^j\zeta(t)dt=\int_0^\infty t^j\varphi(t)\,[1-\alpha(t)]dt.\end{aligned} \tag{2.29}$$

Another quantity of interest that can be derived from the present analysis is the distribution of the number of pedestrians crossing between two consecutive vehicles. Let $\pi_k$ be the probability that $k$ pedestrians cross between two successive vehicles. In solving for $\pi_k$ we must take two cases into account; either the group of pedestrians initially in queue (when the first car passes the group) cross together, or it does not. The probability that $k$ or more pedestrians arrive in a time $t$ (the time delay between the arrival between two successive cars) and that exactly $k$ of these cross, is

$$\sum_{m=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^{k+m}}{(k+m)!}\binom{k+m}{m}[\alpha^*(t)]^k[1-\alpha^*(t)]^m=\frac{[\lambda t\alpha^*(t)]^k}{k!}e^{-\lambda t\alpha^*(t)}=e^{-\lambda(t-T(t))}\frac{[\lambda(t-T(t))]^k}{k!} \tag{2.30}$$

where $T(t)$ is defined in eq (2.3). Hence, considering separately the two cases mentioned above, we have, for the probability that $k$ pedestrians cross assuming that the group initially in queue crosses.

$$\pi_k^{(1)}=\sum_{j=0}^{k}\theta_j(\infty)\int_0^\infty \alpha(t)\varphi(t)\,\frac{[\lambda t\alpha^*(t)]^{k-j}}{(k-j)!}\,e^{-\lambda t\alpha^*(t)}dt. \tag{2.31}$$

In the second case, where it is assumed that the group in queue does not cross we have,

$$\pi_k^{(2)}=\int_0^\infty [1-\alpha(t)]\varphi(t)\frac{[\lambda t\alpha^*(t)]^k}{k!}\,e^{-\lambda t\alpha^*(t)}dt. \tag{2.32}$$

In this situation the size of the initial queue is obviously irrelevant. Finally $\pi_k$ is just the sum

$$\pi_k = \pi_k^{(1)} + \pi_k^{(2)}. \tag{2.33}$$

The moments can be derived from the generating function

$$\pi(s) = \sum_{k=0}^{\infty} \pi_k s^k = \int_0^{\infty} \alpha(t)\varphi(t)\psi_x(s) e^{-\lambda t \alpha^*(t)(1-s)} dt + \int_0^{\infty} [1-\alpha(t)]\varphi(t) e^{-\lambda t \alpha^*(t)(1-s)} dt. \tag{2.34}$$

The first two moments of this distribution, denoted by $\overline{n_1}$ and $\overline{n_2}$ are

$$\overline{n_1} = \mu_1(\infty) \int_0^{\infty} \alpha(t)\varphi(t)dt + \int_0^{\infty} \lambda t \alpha^*(t)\varphi(t)dt = \lambda \int_0^{\infty} t\varphi(t)dt$$

$$\overline{n_2} = \mu_2(\infty) \int_0^{\infty} \alpha(t)\varphi(t)dt + 2\mu_1(\infty) \int_0^{\infty} \lambda t \alpha^*(t)\alpha(t)\varphi(t)dt$$

$$+ \int_0^{\infty} [\lambda t \alpha^*(t)]^2 \varphi(t)dt + \overline{n_1}. \tag{2.35}$$

Similar calculations enable us to find the probability that there are $m$ pedestrians in queue at a random time, not necessarily just after the passage of a car. As before we shall assume that the event "passage of a car" is a regeneration point for the process, and calculate a set of transition probabilities, $\{\lambda_{nm}\}$, where $\lambda_{nm}$ is the probability that there are $m$ pedestrians in queue at a random time, given that there were $n$ in queue at the last regeneration point. For the present calculation we assume that the gap in which the random time finds itself is of duration $t$, and that the last regeneration point occurred $y$ units of time ago, where clearly $y \leq t$.

We first calculate the probability density for the duration of the gap, $t$, in which the instant find: itself. Let $u(x)dx$ be the probability for the occurrence of a regeneration point to occur in $(x, x+dx)$. Let $\rho(t, y)dt$ be the probability that a point at $y$ is covered by a gap whose duration satisfies $t \leq G < t+dt$. We shall only be interested in the limit $y \to \infty$. The probability that any gap has duration $G$ such that $t \leq G < t+dt$ is just $\varphi(t)dt$. However, we must account for the fact that the last regeneration point lies in the interval[3] $y$, $y-t$. This probability is just $\int_{y-t}^{y} u(x)dx$, hence the probability density for the size of gap including a random time point is

$$\rho(t, y) = \varphi(t)\left[\int_0^y u(x)dx - \int_0^{y-t} u(x)dx\right] + 0(1) \tag{2.36}$$

in the limit of $y \to \infty$. Under quite general conditions on $\varphi(t)$ which are always met in practice it can be shown by the methods of renewal theory, [13], that

$$\int_0^y u(x)dx \sim \frac{y}{\nu_1} \tag{2.37}$$

in the limit of $y \to \infty$, where

$$\nu_1 = \int_0^{\infty} t\varphi(t)dt. \tag{2.38}$$

Hence we finally find

$$\rho(t, \infty) = \frac{t\varphi(t)}{\nu_1}. \tag{2.39}$$

This result has been derived in a slightly different manner by Oliver and Jewell, [15].

[3] We tacitly assume that $y$ is chosen so large that $y > t$. A more complete result which does not use this assumption is easy to prove, but in what follows we are only interested in the equilibrium result.

We now define a function $\delta_{nm}(y, t)$ by $\delta_{nm}(y, t)dy$ = probability that there were $n$ pedestrians in queue at the last regeneration point which occurred in $(t-y, t-y+dy)$, and that $m$ pedestrians are presently in queue. The quantities which we seek, the $\lambda_{nm}$ are related to $\delta_{nm}(y, t)$ by

$$\lambda_{nm} = \int_0^\infty \rho(t)dt \int_0^t \frac{dy}{t} \delta_{nm}(y, t) \tag{2.40}$$

since the time of observation is uniformly distributed over the gap duration. An expression for $\delta_{nm}(y, t)$ can immediately be written down by exactly following the argument which led to eq (2.5). The final result is

$$\begin{aligned} \delta_{nm}(y, t) &= \alpha(t)e^{-\lambda T^+(y,\,t)}\frac{[\lambda T^+(y, t)]^n}{n!} \quad n > m \\ \delta_{nm}(y, t) &= \alpha(t)e^{-\lambda T^+(y,\,t)}\frac{[\lambda T^+(y, t)]^n}{n!} \quad m \geq n \\ &\quad + (1-\alpha(t))e^{-\lambda T^+(y,\,t)}\frac{[\lambda T^+(y, t)]^{m-n}}{(m-n)!} \end{aligned} \tag{2.41}$$

where

$$\lambda T^+(y, t) = \lambda \int_{t-y}^t (1-\alpha(u))du. \tag{2.42}$$

Hence, finally, $\lambda_{nm}$ can be expressed as the sum of two terms:

$$\begin{aligned} \lambda_{nm} &= A_m \qquad n > m \\ &= A_m + B_{m-n} \quad m \geq n \end{aligned} \tag{2.43}$$

where, combining eqs (2.38), (2.40), and (2.42)

$$\begin{aligned} A_n &= \frac{1}{n!\nu_1}\int_0^\infty \alpha(t)\varphi(t)dt \int_0^t [\lambda T^+(y, t)]^n e^{-\lambda T^+(y,\,t)}dy \\ B_r &= \frac{1}{r!\nu_1}\int_0^\infty (1-\alpha(t))\varphi(t)dt \int_0^t [\lambda T^+(y, t)]^r e^{-\lambda T^+(y,\,t)}dy. \end{aligned} \tag{2.44}$$

Let us denote by $c_m$ the probability that there are $m$ pedestrians in queue at a random time. This probability is expressible in terms of the $\theta_n(\infty)$ as

$$c_m = \sum_{n=0}^{\infty} \theta_n(\infty)\lambda_{nm} = A_m + \sum_{n=0}^{m} \theta_n(\infty)B_{m-n}. \tag{2.45}$$

Since this equation is in convolution form we can profitably write down the generating function for $c_m$, call it $C(s)$, and relate it to the generating functions of $A_m$ and $B_m$, $A(s)$ and $B(s)$ respectively. In this way we are led to

$$C(s) = A(s) + \frac{\Delta(s)}{1-\epsilon(s)}B(s) \tag{2.46}$$

where we have used eq (2.13) for the generating function of the $\theta_n(\infty)$. The generating functions $A(s)$ and $B(s)$ are explicitly

$$\begin{aligned} A(s) &= \frac{1}{\nu_1}\int_0^\infty \alpha(t)\varphi(t)dt \int_0^t e^{-\lambda T^+(y,\,t)(1-s)}dy \\ B(s) &= \frac{1}{\nu_1}\int_0^\infty (1-\alpha(t))\varphi(t)dt \int_0^t e^{-\lambda T^+(y,\,t)(1-s)}dy. \end{aligned} \tag{2.47}$$

For the first two moments we find by differentiation

$$\overline{m} = \frac{\lambda}{\nu_1}\int_0^\infty \varphi(t)dt \int_0^t T^+(y,t)dy + \frac{\mu_1(\infty)}{\nu_1}\int_0^\infty t(1-\alpha(t))\varphi(t)dt$$

$$\overline{m^2} = \overline{m} + \frac{\lambda^2}{\nu_1}\int_0^\infty \varphi(t)dt \int_0^t [T^+(y,t)]^2 dy$$

$$+ \frac{2\mu_1(\infty)\lambda}{\nu_1}\int_0^\infty (1-\alpha(t))\varphi(t)dt \int_0^t T^+(y,t)dy$$

$$+ \frac{\mu_2(\infty)}{\nu_1}\int_0^\infty t(1-\alpha(t))\varphi(t)dt. \tag{2.48}$$

When $\alpha(t)$ is a step function,

$$\begin{aligned} \alpha(t) &= 0 \qquad t < T_c \\ &= 1 \qquad t > T_c \end{aligned} \tag{2.49}$$

the double integrals can be reduced to single integrals by introducing the explicit form for $T^+(y,t)$:

$$\begin{aligned} T^+(y,t) &= 0 && t > T_c, \quad t-y > T_c \\ &= T_c - t + y && t > T_c, \quad t-y < T_c \\ &= y && t < T_c. \end{aligned} \tag{2.50}$$

When this form for $T^+(y,t)$ is substituted into eq (2.47), we obtain the results

$$\overline{m} = \frac{1}{2\nu_1}\int_0^{T_c} t^2\varphi(t)dt + \frac{\lambda T_c^2}{2\nu_1}\Phi(T_c) + \frac{\mu_1(\infty)}{\nu_1}\int_0^{T_c} t\varphi(t)dt$$

$$\overline{m^2} = \overline{m} + \frac{\lambda^2}{3\nu_1}\int_0^{T_c} t^3\varphi(t)dt + \frac{\lambda^2 T_c^3}{3\nu_1}\Phi(T_c) \tag{2.51}$$

$$+ \frac{\mu_1(\infty)\lambda}{\nu_1}\int_0^{T_c} t^2\varphi(t)dt$$

$$+ \frac{\mu_2(\infty)}{\nu_1}\int^{T_c} t\varphi(t)dt$$

where

$$\Phi(t) = \int_t^\infty \varphi(x)dx. \tag{2.52}$$

## 3. Some Applications

We shall now apply some of the preceding general results to the specific distributions

$$\begin{aligned} \varphi_m(t) &= \sigma e^{-\sigma t}\frac{(\sigma t)^m}{m!} && m = 0, 1, 2, \ldots \\ \alpha(t) &= 0 && t < T_c \\ &= 1 - e^{-\beta(t-T_c)} && t \geq T_c. \end{aligned} \tag{3.1}$$

When $\beta$ tends to infinity this form of $\alpha(t)$ tends to the step function

$$\begin{aligned} \alpha(t) &= 0 \qquad t < T_c \\ &= 1 \qquad t \geq T_c \end{aligned} \tag{3.2}$$

which has been used extensively in previous treatments of this subject. It is easily verified that for this choice of $\alpha(t)$ we have for $\alpha^*(t)$:

$$\begin{aligned}\alpha^*(t) &= \quad 0 \qquad t < T_c \\ &= \frac{1}{\beta t}[\beta t - \beta T_c - 1 + e^{-\beta(t-T_c)}] \qquad t > T_c\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}T(t) &= t[1-\alpha^*(t)] \\ &= t \qquad t < T_c \\ &= \frac{1}{\beta}[1+\beta T_c - e^{-\beta(t-T_c)}] \qquad t \geq T_c\end{aligned} \tag{3.4}$$

A quantity which frequently occurs in the calculations is $\epsilon(1)$, which for the present set of models is

$$\begin{aligned}\epsilon(1) &= \int_0^\infty \varphi_m(t)[1-\alpha(t)]dt \\ &= \sigma \int_0^{T_c} e^{-\sigma t}\frac{(\sigma t)^m}{m!}\,dt + \sigma\int_{T_c}^\infty e^{-\sigma t}\frac{(\sigma t)^n}{n!}\,e^{-\beta(t-T_c)}dt \\ &= 1 - G_m(\sigma T_c) + \left(\frac{\sigma}{\sigma+\beta}\right)^{m+1} e^{\beta T_c} G_m[(\sigma+\beta)T_c]\end{aligned} \tag{3.5}$$

where

$$G_m(x) = e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^m}{m!}\right). \tag{3.6}$$

When $\beta = \infty$, the expression for $\epsilon(1)$ becomes

$$\epsilon(1) = 1 - G_m(\sigma T_c). \tag{3.7}$$

An expression for the quantity

$$\int_0^\infty \lambda T(t)\varphi_m(t)dt$$

which appears in the formula for $\mu_1(\infty)$ (of eq (2.19)) is

$$\begin{aligned}\int_0^\infty \lambda T(t)\varphi_m(t)dt &= (m+1)\frac{\lambda}{\sigma}[1 - G_{m+1}(\sigma T_c)] \\ &\quad + \frac{\lambda}{\beta}(1+\beta T_c)G_m(\sigma T_c) - \frac{\lambda}{\beta}e^{\beta T_c}\left(\frac{\sigma}{\sigma+\beta}\right)^{m+1} G_m[(\sigma+\beta)T_c].\end{aligned} \tag{3.8}$$

When $\beta = \infty$ this expression reduces to

$$\int_0^\infty \lambda T(t)\varphi_m(t)dt = (m+1)\frac{\lambda}{\sigma}[1 - G_{m+1}(\sigma T_c)] + \lambda T_c G_m(\sigma T_c). \tag{3.9}$$

As a specific example of the use of these formulas, let us specialize to the case $m = 0$, i.e.,

$$\varphi(t) = \sigma e^{-\sigma t} \tag{3.10}$$

where $\sigma$ is the expected number of cars passing per unit time. Then it is easily found that

$$\epsilon(1) = 1 - \frac{\beta}{\sigma+\beta}e^{-\sigma T_c} \tag{3.11}$$

and

$$\mu_1(\infty) = \lambda\left(\frac{1}{\sigma} + \frac{1}{\beta}\right)\left(e^{\sigma T_c} - \frac{\beta}{\beta+\sigma}\right) \tag{3.12}$$

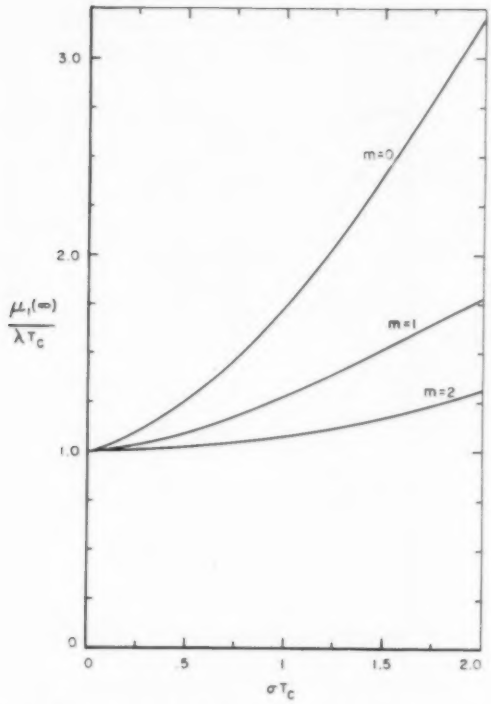


FIGURE 1. *Plot of normalized expected queue length as a function of normalized traffic intensity.*

is the equilibrium value of the expected number of people in queue at a regeneration point. When $\beta=\infty$ we obtain the value given by Tanner for the case of a negative exponential headway distribution:

$$\mu_1(\infty)=\frac{\lambda}{\sigma}(e^{\sigma T_c}-1). \tag{3.13}$$

We find in general for $\beta=\infty$, the value of $\mu_1(\infty)$ is

$$\mu_1(\infty)=\lambda T_c+(m+1)\frac{\lambda}{\sigma}\left[\frac{1-G_{m+1}(\sigma T_c)}{G_m(\sigma T_c)}\right]. \tag{3.14}$$

Several curves showing $\mu_1(\infty)/(\lambda T_c)$, the normalized expected number of pedestrians in queue at the passage of a random car are shown in figure 1. As $\sigma\to\infty$, i.e., the mean gap between vehicles decreases to zero, we find

$$\mu_1(\infty)=\lambda T_c\left[\frac{(m+1)!}{(\sigma T_c)^{m+1}}e^{\sigma T_c}+0(1)\right]. \tag{3.15}$$

When $\sigma\to 0$ we find

$$\mu_1(\infty)\sim\lambda T_c\left[1+\frac{(\sigma T_c)^{m+1}}{m!(m+2)}\right]. \tag{3.16}$$

The equilibrium probability that there are no pedestrians in queue when a random car passes is given in eq (2.24). When $\varphi(t)$ is given by $\varphi_m(t)$ of eq (3.1) and $\alpha(t)$ is a step function $\theta_0(\infty)$ is

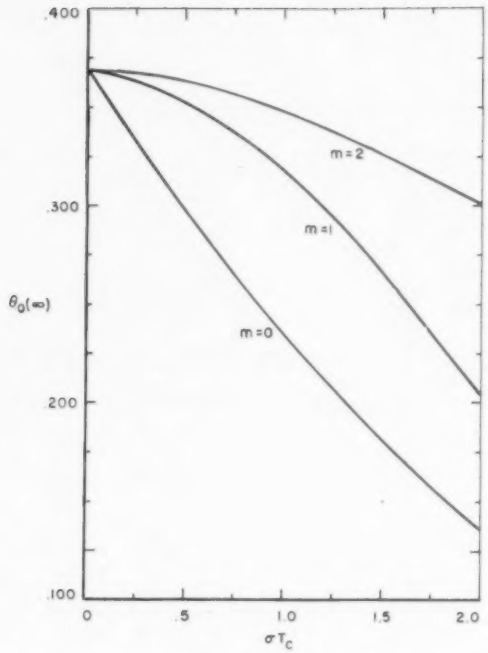


FIGURE 2. *Plot of probability of an empty queue as a function of normalized traffic intensity.*

given by

$$\theta_0(\infty)=\frac{e^{-\lambda T_c}G_m(\sigma T_c)}{1-\left(\frac{\sigma}{\sigma+\lambda}\right)^{m+1}(1-G_m\{(\sigma+\lambda)T_c\})}. \tag{3.17}$$

In particular when $m=0$ this expression reduces to that derived by Tanner, [4],

$$\theta_0(\infty)=\frac{\sigma+\lambda}{\lambda e^{(\sigma+\lambda)T_c}+\sigma}. \tag{3.18}$$

Figure 2 contains plots of $\theta_0(\infty)$ for $\lambda T_c=1$ and different values of $m$. When $\sigma\to\infty$ eq (3.17) yields the approximate result

$$\theta_0(\infty)=\frac{(\sigma T_c)^m}{m!}\,\frac{e^{-(\lambda+\sigma)T_c}}{1-\left(\frac{\sigma}{\sigma+\lambda}\right)^{m+1}}+0[(\sigma T_c)^{m-1}] \tag{3.19}$$

and when $\sigma\to 0$, or equivalently, traffic is light, eq (3.17) reduces to

$$\theta_0(\infty)=e^{-\lambda T_c}\left[1-(\sigma T_c)^{m+1}\left\{\frac{1}{(m+1)!}-\frac{1}{(\lambda T_c)^{m+1}}+\frac{G_m(\lambda T_c)}{(\lambda T_c)^{m+1}}\right\}\right]+0((\sigma T_c)^{m+1}). \tag{3.20}$$

In the first approximation we see that the probability that there are no pedestrians in queue does not depend on the vehicular traffic, but only on the pedestrian arrival rate.

The expected number of pedestrians crossing between two cars can be calculated from eq (2.35) When $\varphi(t)$ is of the form of eq (3.1), then detailed calculation leads to the result

$$\bar{n}_1=(m+1)\frac{\lambda}{\sigma}. \tag{3.21}$$

The expected size of the queue at a random time is given in the first line of eq (2.47). When $\varphi_k(t)$ and $\alpha(t)$ are of the form given in eq (3.1) we find

$$\overline{m}=\frac{\lambda}{2}\,\frac{(k+2)}{\sigma}\,[1-G_{k+2}(\sigma T_c)]+\frac{\lambda\sigma T_c^2}{2(k+1)}\,G_k(\sigma T_c)$$
$$+[1-G_{k+1}(\sigma T_c)]\left[\lambda T_c+(k+1)\frac{\lambda}{\sigma}\,\frac{1-G_{k+1}(\sigma T_c)}{G_k(\sigma T_c)}\right]. \tag{3.22}$$

When $\sigma\to\infty$ this expression becomes approximately

$$\overline{m}=\frac{(k+1)!\lambda e^{\sigma T_c}}{\sigma(\sigma T_c)^k}+0(1) \tag{3.23}$$

and for $\sigma\to 0$ the expression goes into

$$\overline{m}=\lambda T_c[1+0(\sigma T_c)]. \tag{3.24}$$

For the Poisson distribution we set $k=0$ and obtain the exact result

$$\overline{m}=\frac{\lambda}{\sigma}(e^{\sigma T_c}-1-\sigma T_c) \tag{3.25}$$

as derived by Tanner.

## 4. Comments and Generalizations

One of the weakest points of the preceding analysis is the assumption that a group of pedestrians will have the same gap acceptance probability as a single pedestrian. If we assume that there are, instead, an ensemble of gap acceptance probabilities $\{\alpha_n(t)\}$ then the form of $\mathbf{P}$ is not the same as that shown in eq (2.5), and it is necessary to discuss the resulting Markov chain ab initio. In particular it is no longer possible to have as simple a relation between generating functions as that given in eq (2.12). The more complete form for $\mathbf{P}$ is easily shown to be

$$\mathbf{P}=\begin{bmatrix}\Delta_{00}+\epsilon_{00} & \Delta_{01}+\epsilon_{01} & \Delta_{02}+\epsilon_{02} & \cdot\ \cdot\ \cdot\ \cdot\\ \Delta_{10} & \Delta_{11}+\epsilon_{11} & \Delta_{12}+\epsilon_{12} & \cdot\ \cdot\ \cdot\ \cdot\\ \Delta_{20} & \Delta_{21} & \Delta_{22}+\epsilon_{22} & \cdot\ \cdot\ \cdot\ \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{bmatrix} \tag{4.1}$$

where

$$\Delta_{ij}=\int_0^\infty \alpha_i(t)e^{-\lambda t}[1-\alpha_i^*(t)]\,\frac{[\lambda t(1-\alpha_i^*(t))]^j}{j!}\,\varphi(t)dt$$
$$\epsilon_{ij}=\int_0^\infty (1-\alpha_i(t))e^{-\lambda t}[1-\alpha_i^*(t)]\,\frac{[\lambda t(1-\alpha_i^*(t))]^{j-i}}{(j-i)!}\,\varphi(t)dt. \tag{4.2}$$

Attempts to solve the steady state equations corresponding to the transition matrix of eq (4.1) have been unsuccessful. It is no longer clear in this situation that the queue will reach a steady state, although this will evidently be the case when $\alpha_1(t)\leq\alpha_2(t)\leq\alpha_3(t)\leq\ \cdot\ \cdot\ \cdot$ since we have seen that a steady state is reached when all of the $\alpha(t)$ are equal, and the inequality provides for faster servicing of the arriving pedestrians.

There is one case of differential gap acceptance probabilities that can be solved. Let us assume that

$$\alpha_1(t)=\beta(t),\ \alpha_2(t)=\alpha_3(t)=\ .\ .\ .=\alpha(t) \tag{4.3}$$

that is, that group behavior will differ from individual behavior, although the gap acceptance probability for a group of greater than one will not depend on the size of the group. We shall examine the steady state equations only, i.e., the solution to the equation $\boldsymbol{\theta}(\infty)=\boldsymbol{\theta}(\infty)\,\mathbf{P}$. When these equations are written out in component form (omitting the infinity in the argument) we find

$$\theta_n=\Delta_n+\sum_{r=0}^{n}\theta_r\epsilon_{n-r}+\theta_1\gamma_n. \tag{4.4}$$

where $\Delta_j$ and $\epsilon_j$ are just the functions defined in eq (2.6), and

$$\gamma_n=\int_0^\infty \varphi(t)e^{-\lambda T}[\beta(t)-\alpha(t)]\left[\frac{(\lambda T)^n}{n!}-\frac{(\lambda T)^{n-1}}{(n-1)!}\right]dt \tag{4.5}$$

$$n\geq 1$$

$$\gamma_0=\int_0^\infty \varphi(t)e^{-\lambda T}[\beta(t)-\alpha(t)]dt$$

and

$$T(t)=\int_0^t(1-\beta(u))du. \tag{4.6}$$

The equation satisfied by the generating function is

$$\Psi(s)=\frac{\Delta(s)+\theta_1\gamma(s)}{1-\epsilon(s)} \tag{4.7}$$

where $\Psi(s)=\sum_{n=0}^{\infty}\theta_n s^n$ and $\gamma(s)=\sum_{n=0}^{\infty}\gamma_n s^n$. The right side of this equation still contains the unknown value of $\theta_1$, but this can be eliminated by differentiating both sides and setting $s$ equal to zero. With this procedure we find

$$\theta_1=\frac{\Delta_1(1-\epsilon_0)+\Delta_0\epsilon_1}{(1-\epsilon_0)^2-\gamma_1(1-\epsilon_0)-\gamma_0\epsilon_1} \tag{4.8}$$

which can then be substituted back into eq (4.7) to give the full expression for the generating function.

When it is assumed that the first $r$ of the $\alpha_j(t)$ differ, but the remaining $\alpha_j(t)$ are all equal, then $r$ of the $\theta$'s appear in the expression for the generating function. These can be found by inverting an $rxr$ matrix. We will not pursue this line of inquiry any further since no experimental data exists which is relevant to this matter.

Another shortcoming of the model, even in the somewhat more general formulation of the last paragraph, is the fact that either none or all of the pedestrians in queue are assumed to cross at the same time. One can formally remedy this deficiency by defining a matrix $(\alpha_{nr}(t))$ such that $\alpha_{nr}(t)$ is the probability that $r$ out of $n$ pedestrians will accept a gap of duration $t$. However, detailed calculations show that the resultant transition matrix is such that the stationary problem cannot be solved by generating function methods, and so far the solution to the problem is unknown.

It is possible by our methods to treat the queueing problems with bunched arrivals, where the

bunches arrive in accordance with a Poisson process, but the number in each bunch is a random variable with a known distribution. Again, detailed calculation for this situation is hardly informative without any supporting data.

---

I thank Alan Goldman for his careful reading of this manuscript and for several suggestions which have been incorporated into the final draft.

## 5. References

[1] W. F. Adams, Road traffic considered as a random series, Inst. Civil Eng. J. **4,** 121 (1936).
[2] F. Garwood, An application of the theory of probability to the operation of vehicular-controlled traffic signals, J. Roy. Stat. Soc. (Supplement) **7,** 65 (1940).
[3] M. S. Raff, The distribution of blocks in an uncongested stream of automobile traffic, J. Am. Stat. Soc. **46,** 114 (1951).
[4] J. C. Tanner, The delay to pedestrians crossing a road, Biometrika **38,** 383–342 (1951).
[5] A. J. Mayne, Some further results in the theory of pedestrians and road traffic, Biometrika **41,** 375–389 (1954).
[6] G. H. Weiss, and A. A. Maradudin, Some problems in traffic delay, J. Op. Res. **10,** 74 (1962).
[7] R. M. Oliver, and E. F. Bisbee, Queueing for gaps in high flow traffic streams, J. Op. Res. **10,** 105 (1962).
[8] R. M. Oliver, Distribution of gaps and blocks in a traffic stream, J. Op. Res. **10,** 197 (1962).
[9] W. S. Jewell, Waiting for a gap in traffic, Res. Report 6, Operations Research Center, University of California (Berkeley) (1961).
[10] J. Cohen, E. J. Dearnaley, and C. E. M. Hansel, The risk taken in crossing a road, Op. Res. Quart. **6,** 120 (1955).
[11] P. A. Perchonok, and S. L. Levy, Applications of digital simulation techniques to freeway on-ramp traffic operations, Highway Research Board Proceedings, 506 (1960).
[12] R. Herman, and G. H. Weiss, Comments on the highway-crossing problem, J. Op. Res. **9,** 828 (1961).
[13] W. L. Smith, Renewal theory and its ramifications, J. Roy. Stat. Soc. **B20,** 243 (1958).
[14] L. Takács, Introduction to the theory of queues, (Oxford Press, New York, 1962).
[15] W. S. Jewell, and R. M. Oliver, The distribution of spread, Res. Report 20, Operations Research Center, University of California (Berkeley) (1961).

(Paper 67B4–105)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 4, October–December 1963

# Solutions of the Equation $\Psi_{xx} + \frac{1}{x}\Psi_x + Kx^n e^{\Psi} = 0$

Elliot A. Kearsley

(July 31, 1963)

The general solution of the equation $\Psi_{xx} + \frac{1}{x}\Psi_x + Kx^n e^{\Psi} = 0$ is displayed in terms of simple tabulated functions. The existence and uniqueness of solutions of a simple boundary value problem are determined as a function of the parameter $K$.

For a class of problems in the steady flow of viscous fluids with a viscosity depending exponentially on temperature, an ordinary, nonlinear, second-order, differential equation arises [1,2][1]; viz,

$$\Psi_{xx} + \frac{1}{x}\Psi_x + Kx^n e^{\Psi} = 0;\ x > 0. \tag{1}$$

The same equation may arise in calculating the temperature distribution in a dielectric in an alternating field [3]. In this equation, $\Psi$ is an unknown function of $x$, while $K$ and $n$ are arbitrary real constants. The equation is of considerable mathematical interest as an example of that exceedingly small class of strongly nonlinear equations whose general solution can be displayed in terms of simple tabulated functions.

The reader may verify by direct differentiation that the following array is indeed a collection of solutions of eq (1):

$$\Psi = \log\left\{\frac{2m^2}{Kx^{n+2}\cosh^2(m\log x + a)}\right\},\ K > 0. \tag{2a}$$

$$\left.\begin{aligned}
\Psi &= \log\left\{\frac{-2m^2}{Kx^{n+2}\sinh^2(m\log x + a)}\right\}, && (2b')\\
\Psi &= \log\left\{\frac{-2m^2}{Kx^{n+2}\sin^2(m\log x + a)}\right\}, && (2b'')\\
\Psi &= \log\left\{\frac{-2}{Kx^{n+2}(\log x + a)^2}\right\}, && (2b''')
\end{aligned}\right\} K < 0$$

$$\Psi = m\log x + a,\ K = 0. \tag{2c}$$

In these equations, $m$ and $a$ are two arbitrary real constants which may be thought of as constants of integration.

It is important to show that this array contains all the real solutions of eq (1). One can get at this question through some well known properties of ordinary differential equations. Observe that eq (1) can be written in the form of a system of two first order equations:

$$\Psi_x = \varphi = f_1(\varphi, \Psi, x)$$

$$\varphi_x = -\frac{\varphi}{x} - Kx^n e^{\Psi} = f_2(\varphi, \Psi, x) \tag{3}$$

It is immediately clear that the partial derivatives of $f_1$ and $f_2$ are continuous in the region $x > 0$, and hence, by a well-known theorem [4] eq (1) satisfies a Lipschitz condition in some neighborhood of each point in the region. It follows immediately [5] that there is exactly one solution of eq (1) with a value and slope assigned at any given value of $x \neq 0$. We can thus be sure that the array of eq (2) contains all the solutions if we can find values of the constants of integration which determine a solution of preassigned value and preassigned derivative for arbitrary, nonnegative $x$. We notice that eq (1) may be transformed in the following way. Let $\Psi^* = \Psi + \Psi_0$ and $y = x/x_0$, then eq (1) becomes

$$\Psi^*_{yy} + \frac{1}{y}\Psi^*_y + Kx_0^{n+2}e^{-\Psi_0}y^n e^{\Psi^*} = 0. \tag{4}$$

This equation is of the same form as eq (1) with $K$ replaced by $Kx_0^{n+2}e^{-\Psi_0}$, while $\Psi_0$ and $x_0$ are unspecified constants. We see, therefore, that it is sufficient to show that among the solutions of eq (1) which vanish at $x = 1$, there is a solution whose derivative at $x = 1$ takes on any preassigned value.

Consider eq (2a) evaluated at $x = 1$. By the vanishing of $\Psi$, we have that $m = \pm\sqrt{\frac{K}{2}}\cosh a$, and thus that

$$(\Psi_x)_{x=1} = -(n+2) \pm 2\sqrt{\frac{K}{2}}\sinh a.$$

It is clear that we may choose $a$ so as to make $(\Psi_x)_{x=1}$ take on any real finite value.

[1] Figures in brackets indicate the literature references at the end of this paper.

When $K<0$, any one of the three forms of the solution, 2b′, 2b″, and 2b‴ is possible. By the vanishing of $\Psi$ at $x=1$ we have respectively, $m=\pm\sqrt{\frac{-K}{2}}\sinh a$, $m=\pm\sqrt{\frac{-K}{2}}\sin a$ and $a=\pm\sqrt{\frac{2}{-K}}$. The slope of $\Psi$ at $x=1$ can then be seen to be,

$$(\Psi_x)_{x=1}=-(n+2)\pm 2\sqrt{\frac{-K}{2}}\cosh a, \qquad a\neq 0$$

$$(\Psi_x)_{x=1}=-(n+2)\pm 2\sqrt{\frac{-K}{2}}\cos a, \qquad a\neq 0$$

$$(\Psi_x)_{x=1}=-(n+2)\pm 2\sqrt{\frac{-K}{2}}$$

for the three forms 2b′, 2b″, and 2b‴, respectively. It is again clear that we may choose $a$ in these equations so as to make $(\Psi_x)_{x=1}$ take on any real finite value. We see further that all three forms are necessary, indeed, that we must choose form 2b′, 2b″ or 2b‴ according to whether the absolute value of $(\Psi_x)_{x=1}+(n+2)$ is greater than, less than, or equal to $2\sqrt{\frac{-K}{2}}$.

For the case $K=0$, the solution must be of the form (2c), but since eq (1) is linear when $K=0$, the existence and uniqueness of solutions is well known and needs no further discussion.

We have thus shown that the array of forms of eq (2) contains all real solutions of eq (1) in the region $x>0$.

Of practical interest is the question of existence and uniqueness of solutions of boundary value problems. We shall consider a boundary value problem in which values of $\Psi$ are given at two values of $x$ and a solution of eq (1) is sought which is bounded and smooth in the interval. Assuming that two solutions exist, call them $\Psi_1$ and $\Psi_2$, we define the difference $\Psi_1-\Psi_2$ as a new function $\omega$. We then perform the following integration by parts,

$$\int_{x_0}^{x_1}\omega\frac{\partial}{\partial x}(x\omega_x)dx=x\omega\omega_x\Big|_{x_0}^{x_1}-\int_{x_0}^{x_1}x\omega_x{}^2dx, \tag{5}$$

where $x_0$ and $x_1$ are the values of $x$ at the boundaries of the interval. We replace the integral on the left of this equation by using the fact that both $\Psi_1$ and $\Psi_2$ satisfy eq (1) to get

$$K\int_{x_0}^{x_1}x^{n+1}e^{\Psi_2}\omega(1-e^{\omega})dx=x\omega\omega_x\Big|_{x_0}^{x_1}-\int_{x_0}^{x_1}x\omega_x{}^2dx. \tag{6}$$

Now it is very easy to see that the integrand on the left is always negative. Furthermore, if we restrict ourselves to solutions with bounded derivatives it is clear that the first term on the right vanishes (since $\omega$ vanishes on the boundaries) and, since the integral on the right is always positive, the right hand side of this equation is always negative. Hence, *if $K$ is negative*, there is at most one solution to the boundary value problem. *If $K$ is greater than zero* this uniqueness proof breaks down, and indeed we shall see that multiple solutions may be possible.

We will next consider the case when $K<0$, and show that there always exists a solution of the boundary value problem. We first use eq (4) to redefine $K$ in terms of the given boundary data so that the vanishing of the scaled $\Psi$ at $x=1$ corresponds to the boundary condition at the first boundary. When this condition is satisfied eq (2b) becomes

$$\left.\begin{aligned}\Psi&=\log\left\{\frac{\sinh^2 a}{x^{n+2}\sinh^2\left(a\pm\sqrt{\frac{-K}{2}}\log x\sinh a\right)}\right\} &(7a)\\ \Psi&=\log\left\{\frac{\sin^2 a}{x^{n+2}\sin^2\left(a\pm\sqrt{\frac{-K}{2}}\log x\sin a\right)}\right\} &(7b)\\ \Psi&=\log\left\{\frac{1}{x^{n+2}\left(\sqrt{\frac{-K}{2}}\log x\pm 1\right)^2}\right\} &(7c)\end{aligned}\right.$$

If the second boundary condition is $\Psi=\Psi_2$ at $x=x_2$, $(x_2>1)$, we shall show that there is always one form of eq (7) which will satisfy the condition and which will define a *smooth bounded function in the interval*. For convenience, we define $\varphi=\exp\frac{1}{2}(\Psi_2+(n+2)\log x_2)$. Then $\varphi$ is a number determined by the boundary data which can be expressed in terms of the constant of integration $a$. From eq (7) we see that there are five possible forms of $\varphi$. From (7a) we get two forms[2]

$$\varphi_1=\frac{\sinh a}{\sinh\left(a+\sqrt{\frac{-K}{2}}\log x_2\sinh a\right)},\ 0<a<\infty,$$

$$\varphi_2=\frac{\sinh a}{\sinh\left(a-\sqrt{\frac{-K}{2}}\log x_2\sinh a\right)},\ 0<a<\alpha,$$

from (7b) we get only one independent form[3]

$$\varphi_3=\frac{\sin a}{\sin\left(a+\sqrt{\frac{-K}{2}}\log x_2\sin a\right)},\ 0<a<\pi,$$

[2] The symbol $\alpha$ here represents the positive root of the equation

$$\frac{\sinh a}{a}=1/\sqrt{\frac{-K}{2}}\log x_2.$$

[3] The form with the negative sign is equivalent to this form with $a$ replaced by $a+\pi$. On the other hand $\varphi_3(2\pi-a)=\varphi_3(a)$, hence it is only necessary to consider values of $a$ between zero and $\pi$.

and from (7c) we get two more forms

$$\varphi_4 = \frac{1}{\sqrt{\frac{-K}{2}}\log x_2 + 1}$$

$$\varphi_5 = \frac{1}{\sqrt{\frac{-K}{2}}\log x_2 - 1}$$

We confine our considerations, at first, to the situation when $\sqrt{\frac{-K}{2}}\log x_2 < 1$. A straightforward study of the range of values of the various forms of $\varphi$ reveals the following scheme:

$$0 < \varphi_1 < \frac{1}{1+\sqrt{\frac{-K}{2}}\log x_2} < \varphi_3 < \frac{1}{1-\sqrt{\frac{-K}{2}}\log x_2} < \varphi_2 < \infty$$

in which, for a suitable choice of $a$ and of the form of $\varphi$, $\varphi$ can take on any value between zero and infinity. It is clear, therefore, that there is always a solution to the boundary value problem.

When $\sqrt{\frac{-K}{2}}\log x_2 \geq 1$ some of the solutions represented by eq (7) are no longer suitable because they no longer correspond to smooth bounded functions in the interval. In particular, forms $\varphi_2$ and $\varphi_5$ correspond to inadmissible solutions, and in form $\varphi_3$, $a$ must be restricted to the range $0 < a < \pi - \beta$, where $\beta$ is the smallest positive root of the equation

$$\frac{\sin\beta}{\beta} = 1/\sqrt{\frac{-K}{2}}\log x_2.$$

We then find the following scheme for the possible values of the various forms of $\varphi$:

$$0 < \varphi_1 < \frac{1}{1+\sqrt{\frac{-K}{2}}\log x_2} < \varphi_3 < \infty$$

and again $\varphi$ can take on any value between zero and infinity, so that there is always a solution to the boundary value problem.

When $K > 0$, we have seen that the uniqueness proof fails. In fact we cannot expect a unique solution of the boundary value problem. Upon fitting the boundary condition $\Psi = 0$ at $x = 1$, eq (2a) becomes

$$\Psi = \log\left\{\frac{\cosh^2 a}{x^{n+2}\cosh^2(a+\sqrt{\frac{-K}{2}}\log x\cosh a)}\right\}, -\infty < a < \infty$$

Again, we represent the second boundary condition by $\Psi = \Psi_2$ at $x = x_2$, $(x_2 > 1)$. As before, we define $\varphi = \exp\frac{1}{2}(\Psi_2 + (n+2)\log x_2)$, and find

$$\varphi = \frac{\cosh a}{\cosh(a+\sqrt{\frac{K}{2}}\log x_2\cosh a).} \tag{8}$$

A study of this function shows it to be a smooth positive function of $a$, vanishing as $a$ approaches plus or minus infinity and with a single maximum at $a = \alpha$, where $\alpha$ is the root of the equation

$$\tanh(\alpha+\sqrt{\frac{K}{2}}\log x_2\cosh\alpha) = \frac{\tanh\alpha}{1+\sqrt{\frac{K}{2}}\log x_2\sinh\alpha}. \tag{9}$$

In this case, therefore, there are two solutions to the boundary value problem when $\varphi < \varphi_{max}$, one solution when $\varphi = \varphi_{max}$ and no solutions when $\varphi > \varphi_{max}$. This non-existence of solutions of the boundary-value problem can be associated with an instability phenomenon in physical problems of viscous heating [6].

## References


[1] E. A. Kearsley, The viscous heating correction for viscometer flows, Trans. Soc. Rheology VI, 253–261.
[2] E. A. Kearsley, Viscous heating in couette flow (in preparation).
[3] C. Copple, D. R. Hartree, A. Porter, and H. Tyson, The evaluation of transient temperature distributions in a dielectric in an alternating field, Instn. of Elect. Engineers 85, 56–66 (1939).
[4] E. Kamke, Differentialgleichungen Reeler Funktionen, p. 124 (Chelsea Publishing Co., New York, 1947).
[5] Ibid., p. 141, Satz 4.
[6] I. J. Gruntfest, Temperatures generated by the flow of liquids in pipes, to appear in J. Appl. Phys. (1963).


(Paper 67B4–106)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 4, October–December 1963

# On the Graphs of Finite Idempotent Boolean Relation Matrices

David Rosenblatt

(July 16, 1963)

This paper presents a graph-theoretic characterization of idempotent Boolean relation matrices of finite order. A relation-theoretic point of view is adopted in the paper. Idempotent matrices appear in the sequence of powers of any Boolean relation matrix, and are of purely theoretical as well as applied interest in connection with issues of convergence. The results provide a detailed description of the connectivity and cyclic structure of the directed graphs of idempotent matrices. The study is basically motivated by certain connectivity and flow problems which arise in the analysis of large-scale information systems. The formal results are exemplified in an investigation of the asymptotic forms of a recursive model of an information system which affords a conjoint representation of processes of communication and derivation of information. A second principal application is given in a process formulation for the generation of consistent rank orderings. The relation between system design and idempotent forms is exhibited in the two applications.

## 1. Introduction

In this paper we present a graph-theoretic characterization of idempotent Boolean relation matrices of finite order, that is, matrices $S$ such that $S^2=S$. We adopt an approach which corresponds to that of an earlier study of the graphs and powers of finite relation matrices and stochastic matrices [9].[1] The present results essentially treat issues of detailed matrix structure.

This study is basically motivated by certain connectivity and flow problems which arise in the analysis of the structure of large-scale information systems. The formal results are exemplified in part in an investigation of the asymptotic forms of a recursive model of an information system [8, 10] (sec. 5). Additional applications are noted in the body of the paper, including among others, a Boolean process formulation of consistent rank orderings [2] (sec. 4).

## 2. Definitions

We adhere essentially to a relation-theoretic point of view in the analysis of the directed graphs of Boolean relation matrices [5]. The terminology given below corresponds in the main to that of some earlier studies [9, 10] and is also consistent with certain current expositions of the theory of graphs [4].

A homogeneous binary or dyadic relation defined on a finite set $\Sigma$ of elements is construed in classical fashion as any rule $\rho$ which specifies for each ordered couple $(a_i, a_j)$ of elements of $\Sigma$ that either the relation $\rho$ obtains between $a_i$ and $a_j$ (symbolically $a_i \rho a_j$) or that it does not obtain (symbolically $a_i \overline{\rho} a_j$). The one-one representation of binary relations $\rho$ on $\Sigma$ by square Boolean matrices $R=\|r_{ij}\|$ $(i,j=1, \ldots, n)$ is defined in the following way: $r_{ij}=1$ if $a_i \rho a_j$; $r_{ij}=0$ if $a_i \overline{\rho} a_j$.

The graph-theoretic representation of binary relations $\rho$ on $\Sigma$ may be usefully stated in terms of the one-one representation of Boolean relation matrices by finite directed graphs. Given any square Boolean relation matrix $R$ of order $n$, the *graph of* $R$, $G(R)$, consists of $n$ objects $\alpha_1, \ldots, \alpha_n$ called *vertices* and the totality of *ordered pairs* of vertices $\overrightarrow{\alpha_i, \alpha_j}$, such that $\overrightarrow{\alpha_i, \alpha_j}$ exists in $G(R)$ if, and only if, $r_{ij}=1$ in $R$. The ordered pair or *edge* $\overrightarrow{\alpha_i, \alpha_j}$ is represented by an arrowed line directed from $\alpha_i$ to $\alpha_j$ with arrowhead pointing toward $\alpha_j$; an edge of the reflexive form $\overrightarrow{\alpha_i, \alpha_i}$ is taken to be admissible for any vertex $\alpha_i$ in $G(R)$ and is represented by a simple loop from $\alpha_i$ back to $\alpha_i$. A *subgraph* of an arbitrary graph $G$ is a subset of the edges and vertices of $G$ containing with each edge its endpoints. Given this one-one representation, it is then useful to consider the Boolean relation matrix $R(G)$ corresponding to any specified directed graph $G$. Any given subgraph $H$ of $G$ $(H \subset G)$ may then be represented by the submatrix $R(H)$ (in the general sense of subrelation) of the relation matrix $R(G)$ corresponding to $G$.

We consider next a sequence of definitions which provide the basis for a primary classification of the directed graphs of homogeneous binary relations defined on finite sets of elements. The fundamental notion employed here is that of *connectivity* [4].

A vertex $\alpha$ of graph $G$ is said to be *connected* to a vertex $\beta$ in a subgraph $H \subset G$ if, and only if, $H$ contains edges $\overrightarrow{\alpha, \gamma_1}, \overrightarrow{\gamma_1, \gamma_2}, \ldots, \overrightarrow{\gamma_{m-1}, \gamma_m}$ and $\gamma_m=\beta$. It is expedient, in this context, to say that in $G$, $\beta$ *is*

[1] Figures in brackets indicate the literature references at the end of this paper.

*attainable or accessible from* $\alpha$ *by means of a directed path of length* m *steps.* In a graph $G$ with subgraphs $H, K$, $H$ is said to be *connected* to $K$, if some vertex of $K$ is attainable from a vertex of $H$. $H$ is said to be *strongly connected* to $K$, if every vertex of $K$ is attainable from a single vertex of $H$.

We consider next the concept of "cyclic net." A subgraph $H \subset G$ is said to be a *cyclic net of order m* if, and only if, $H$ contains $m\,(m > 0)$ vertices of $G$ and each vertex of $H$ is connected (in $H$) to every vertex of $H$. A cyclic net $H$ of order $m$ in graph $G$ is said to be *simple* if, and only if, no proper subgraph $K \subset H$ is a cyclic net. A simple cyclic net will also be called a *cycle.* A *cyclic traverse* is said to exist from a vertex $\alpha$ to a vertex $\beta$ in graph $G$ if, and only if, $\beta$ is accessible from $\alpha$ by means of a directed path with at least one path-vertex contained in a cycle of $G$. An edge in $G$ is said to be *cyclic* if it is contained in some cycle of $G$ and is otherwise *acyclic.*

A cyclic net $H$ of order $m$ in graph $G$ is said to be *maximal in* $G$ if, and only if, every cyclic net in $G$ is a subgraph of $H$ or contains no vertex in common with $H$. A cyclic net $H$ of order $m$ in graph $G$ is said to be *closed in* $G$ if, and only if, $H$ is maximal in $G$ and every vertex of $G$ attainable from any vertex in $H$ is contained in $H$. A cyclic net $H$ of order $m$ in graph $G$ is said to be *universal* if for some positive integer $\hat{q}$ every vertex of $H$ is accessible in $\hat{q}$ steps from some (fixed) vertex $\alpha$ in $H$. A cyclic net which is universal will be called a *universal net*; a universal net is clearly not necessarily maximal in $G$. A cyclic net $H$ is thus universal if, and only if, there is a positive integer $q_0$ such that for all $q \geq q_0$, each vertex of $H$ is attainable from every vertex of $H$ in $q$ steps.

Two properties of universal nets which are of interest in the sequel are the following [9]. A cyclic net is universal if, and only if, the greatest common divisor of the orders of all cycles contained therein is unity. A cyclic net is then at once simple and universal if, and only if, it is of unit order.

## 3. Idempotence

For Boolean relation matrices of finite order, it is clear that idempotence is intrinsically connected with the issue of convergence of matrix powers (cf. [1, 3, 7, 9]). Thus, if $R^*$ is in some sense the unique limit matrix in the sequence of powers of a relation matrix $R$, $R^*$ is idempotent.

For the purpose of this paper, we employ the following notation: $R, S$ square Boolean relation matrices of order $n$; *union*, $R \cup S$; *intersection*, $R \cap S$; *inclusion*, $R \leq S$; *identity*, $R = S$, for $R \leq S$ and $S \leq R$; *proper inclusion*, $R < S$, for $R \leq S$ and $R \neq S$; *relative product or matrix product*, $RS$; *converse* (or *transpose*), $\breve{R}$; *negation* (or *complement*), $\bar{R}$. This is also the scheme of notation employed in connection with binary relations and rectangular Boolean relation matrices.

In the following, a Boolean relation matrix $R$ will be said to be *convergent* in its powers if, and only if, there exists in the sequence $\{R^k; k = 1, 2, \ldots\}$ a power $R^m$ such that $R^m = R^{m+1}$. A relation matrix $R$ will be said to be *oscillatory* or *periodic* in its powers if, and only if, there exists in the sequence $\{R^k; k = 1, 2, \ldots\}$ a power $R^m$ such that $R^m = R^{m+p}$ where $p$ is the smallest integer for which this holds and $p > 1$. Any matrix $R'$ which appears infinitely often in the sequence of powers of a Boolean relation matrix $R$ will be called a *limit matrix* of $R$.

For Boolean relation matrices of finite order, the following two results are immediate [9]. First, a relation matrix is finitely either convergent or oscillatory in its powers. Second, a convergent relation matrix $R$ of order $n(n \geq 2)$ converges to one of the following: (i) The *null matrix* $\Lambda_n = \|r_{ij}\|$, $r_{ij} = 0$ for all $i, j$; (ii) the *universal matrix* $V_n = \|r_{ij}\|$, $r_{ij} = 1$ for all $i, j$; (iii) some idempotent matrix $R^*$ such that $\Lambda_n < R^* < V_n$.

It may be shown that there always exist for any Boolean relation matrix $R$ of order $n$ (periodic or convergent) three idempotent relation matrices which are functions of the powers of $R$. Let $[R]$ denote the set of limit matrices in the sequence of powers of $R$ and let $p \leq n$ denote the cardinal number or period of $[R]$. The elements $R_\alpha^*$ $(\alpha = 1, \ldots, p)$ of $[R]$ constitute a group under the operation of Boolean matrix multiplication with some distinguished one of these (say) $R_1^*$ the group identity [9]. The identity element $R_1^*$ is clearly idempotent. Let $R^{**}$ denote the union of the group elements, $R^{**} = \bigcup_{\alpha=1}^{p} R_\alpha^*$. $R^{**}$ is idempotent. If $R$ is convergent, then $[R]$ is evidently a unit set. Finally, the *transitive closure* (or "ancestral") of $R$,

$$R_* = \bigcup_{h=0}^{\infty} R^h = \bigcup_{h=0}^{i-1} R^h$$

is idempotent; here $R^0 \equiv I_n$, the identity relation matrix of order $n$. The three idempotent relation matrices satisfy the relation $R_1^* \leq R^{**} \leq R_*$ and are in general distinct (see Corollary 2a). If $R$ is convergent, there are at most two distinct matrices. If $I_n \leq R$ so that $R$ is convergent, the three matrices coincide identically. The preceding idempotent forms are of interest in connection with the recursive model of an information system (sec. 5).

The conditions for the convergence of a Boolean relation matrix of order $n(n \geq 2)$ are stated below in a lemma which is given without proof; this result constitutes a reformulation of Theorem 2 of [9].

LEMMA 1: *A Boolean relation matrix* R *of order* n(n ≥ 2) *converges in its powers if, and only if, either the graph* G(R) *contains no cycles or every maximal cyclic net in* G(R) *is universal.*

In a Boolean relation matrix $R = \|r_{ij}\|$ $(i, j = 1, \ldots, n)$ a matrix element $r_{ij}$ will be said to be *1-convergent* (*0-convergent*) in the powers of $R$ if $r_{ij(\alpha)}^* = 1$ (0) in all limit matrices $R_\alpha^*$ $(\alpha = 1, \ldots, p)$ of $R$. An element $r_{ij}$ is said to be *oscillatory* in the powers of $R$ if, and only if, it is neither 1-convergent nor 0-convergent. In the general case, a matrix element may be 1-convergent or 0-convergent even though $R$ is periodic in its powers. For convenience, an element $r_{ij}$ which is not 0-convergent will be called *recurrent* in the powers of $R$. The following lemmas provide a detailed account of

these properties; since these results are essentially obvious we omit proofs.

LEMMA 2a: *An element* $r_{ij}$ *of* R (i, j = 1, . . . , n) *is 0-convergent in the powers of* R *if, and only if, there exists no cyclic traverse from vertex* $\alpha_i$ *to vertex* $\alpha_j$ *in* G(R).

LEMMA 2b: *An element* $r_{ij}$ *of* R (i, j = 1, . . . , n) *is recurrent in the powers of* R *if, and only if, there exist one or more cyclic traverses from vertex* $\alpha_i$ *to vertex* $\alpha_j$ *in* G(R).

LEMMA 2c: *A recurrent element* $r_{ij}$ *of* R *is 1-convergent in the powers of* R *if, and only if, for some positive integer* $q_0$ *the cyclic traverses from vertex* $\alpha_i$ *to vertex* $\alpha_j$ *in* G(R) *exhibit paths of all integral lengths* $q \geq q_0$.

It may be remarked that an element $r_{ij}$ of $R$ is 1-convergent in the powers of $R$ only if the cyclic traverses from vertex $\alpha_i$ to vertex $\alpha_j$ exhibit paths of all integer lengths $q \leq k \pmod{k}$ where $k$ is some cycle order of $G(R)$.

THEOREM 1: *Let* R *be a convergent Boolean relation matrix of order* n. *An element* $r_{ij}$ *of* R *is 1-convergent (0-convergent) if, and only if, there exists a (no) cyclic traverse from vertex* $\alpha_i$ *to vertex* $\alpha_j$ *in the graph* G(R).

PROOF: The result follows directly from Lemmas 1, 2a, and 2c, since 1-convergence coincides with recurrence in this case.

COROLLARY: *Let* R *be a convergent Boolean relation matrix of order* n *with limit matrix* $R^* \neq \Lambda_n$. $R^*$ *is an equivalence relation matrix if, and only if, all vertices and edges of* G(R) *are contained in cycles.*

PROOF: $R^*$ is an equivalence relation matrix if, and only if, it is at once reflexive, symmetric, and transitive. The result then follows directly from the Theorem, since symmetry of $R^*$ requires the mutual inaccessibility of vertices in distinct maximal cyclic nets of $G(R)$.

We consider next a graph-theoretic characterization of the structure of idempotent relation matrices. For this purpose we complete the primary classification of vertices of directed graphs begun earlier. A vertex $\alpha$ in a graph $G$ will be said to be *circuit free in* $G$ if, and only if, $\alpha$ is contained in no cycle of $G$. We proceed to a classification of circuit free vertices.

A subgraph $J \subset G$ is a *null net of order* $m$ in graph $G$ if, and only if, $J$ contains $m (m > 0)$ vertices circuit free in $G$ and all vertices of $J$ are connected to the same vertices of $G$. Obviously, no vertex in a null net $J$ is connected to any vertex of $J$. A null net $J$ of order $m$ in graph $G$ is said to be *maximal in* $G$ if, and only if, every null net in $G$ is a subgraph of $J$ or contains no vertex in common with $J$. A maximal cyclic net or a maximal null net of a graph $G$ will be called a *maximal net* of $G$. Every vertex of a graph $G$ is, therefore, uniquely assigned to some maximal net of $G$.

A maximal null net $N \subset G$ is said to be an *articulated net in* $G$ if, and only if, the following condition obtains: for every vertex $\alpha$ accessible from $N$ in $G$ there exists a cyclic traverse from $N$ to $\alpha$ in $G$. In the graph of a strong ordering relation matrix, for example, the terminal vertex constitutes an articulated net and all other vertices are in null nets of unit order which are not articulated. The concept of maximal null net is evidently important in the formal representation of "redundancy" in graphs or networks.

A subgraph $H \subset G$ is said to be *complete in* $G$ (or a *complete subgraph* of $G$) if, and only if, any vertex of $G$ which is attainable from any vertex of $H$ is attainable in one step. The *null graph* $G$ ($\Lambda_n$) is a complete articulated net; the *universal graph* $G$ ($V_n$) is the unique complete universal net of order $n$.

A graph $G$ will be called a *permanent graph* if, and only if, any maximal net of $G$ is either a complete articulated net or a complete universal net. A permanent graph $G$ is then, evidently, complete.

THEOREM 2: *A Boolean relation matrix* R *is idempotent if, and only if,* G(R) *is a permanent graph.*

PROOF: *Sufficiency.* Assume $G(R)$ is a permanent graph. We show that $R$ is transitive ($R^2 \leq R$) and compact ($R \leq R^2$). $G(R)$ is complete so that $R^2 \leq R$ follows directly. Thus, any pair of vertices $\alpha_i$, $\alpha_j$ of $G(R)$ are either connected in one step or not connected at all. To establish $R \leq R^2$, we show that if any $\alpha_i$ is connected to an $\alpha_j$, then it is connected to it in two and indeed any integral number of steps. Assume that $\alpha_i$ is connected to $\alpha_j$. Then either there exists a cyclic traverse from $\alpha_i$ to $\alpha_j$ in $G(R)$ in which one or both vertices are in universal nets and, consequently (since $G(R)$ is complete), in cycles of unit order, or else both vertices are in articulated nets with an intervening vertex of a universal net between them. In both cases, two-step connection (in fact $k$-step connection, $k \geq 2$) obtains. $R^2 \leq R$ and $R$ is idempotent.

*Necessity.* Assume $R$ is idempotent. Since $R^2 \leq R$, $G(R)$ is complete. By Lemma 1, any maximal cyclic net of $G(R)$ is a complete universal net. Consider any maximal null net $N$ which is connected to any vertex $\alpha$ of a maximal (null or cyclic) net $L$ in $G(R)$. By Theorem 1, there exists a cyclic traverse from $N$ to $\alpha$ of $L$ so that $N$ is a complete articulated net. $G(R)$ is, therefore, a permanent graph.

Theorem 2, in effect, states that in the graph $G(R)$ of an idempotent relation matrix $R$ a vertex $\alpha$ is connected to a vertex $\beta$ if, and only if, $\alpha$ is connected to $\beta$ in one step and there exists a cyclic traverse from $\alpha$ to $\beta$ which passes through a universal net.

COROLLARY 2a: *For any Boolean relation matrix* R *of order* n, *the group union* $R^{**}$ *and the transitive closure* $R_*$ *coincide if, and only if, every vertex of* G(R) *is in a cycle.*

PROOF: The idempotent matrices $R^{**}$ and $R_*$ both have permanent graphs. $G(R_*)$ contains all vertex accessibilities of $G(R)$ (and also slings or auto-accessibilities for all vertices), while $G(R^{**})$ exhibits all recurrent vertex accessibilities of $G(R)$. Thus, $R_* = R^{**}$ if, and only if, all vertex accessibilities of $G(R)$ are recurrent and, in fact coincide with 1-convergences of elements of $R_*$ and $R^{**}$. But this is so if, and only if, every vertex of $G(R)$ is in a cycle.

COROLLARY 2b: *For any Boolean relation matrix* R *of order* n, *there exists a unique permanent graph in the sequence of graphs* $\{G(R^k); k = 1, 2, \ldots\}$.

PROOF: The sequence, in fact, contains at most $(n-1)^2 + 1$ distinct graphs (cf. [9]). The result follows from the Theorem and the unique idempotence of the

identity element in any group and thus in the group of limit matrices of $R$.

To any subgraph $H$ of $G$ there corresponds a unique *complementary graph* $\overline{H}$ (called the *complement of* $H$) consisting of all edges in $G$ which are not in $H$. The unique subgraph of $G$ containing all the vertices of $H$ and no other, and all edges of $\overline{H}$ with both endpoints in $H$ will be called the *proper complementary graph* $\overline{H}'$ (or the *proper complement* of $H$). For any Boolean relation matrix $R$, the proper complement of $G(R)$ is identified with the graph $G(\overline{R})$ of the negation $\overline{R}$ of $R$.

A set of maximal nets in a graph $G$ is said to be a *chain of maximal nets* if, and only if, any two distinct maximal nets in the set are strongly connected. A chain of maximal nets $H_j (j = 1, \ldots, d)$ may then be written in the form $H_1 \succ H_2 \succ \ldots \succ H_d$, where the relation $\succ$ is asymmetric, transitive, and connected in the usual relation-theoretic sense.

We have the following result on complementary permanent graphs.

THEOREM 3: *A graph* G(R) *and its complement are both permanent graphs if, and only if,* G(R) *consists of a chain of complete maximal nets which are alternately universal and articulated.*

PROOF: *Sufficiency.* We show that the condition of the Theorem holds for $G(\overline{R})$ if it holds for $G(R)$. Let $M$ be a maximal null (cyclic) net of $G(R)$. Then the proper complement $\overline{M}'$ of $M$ is a maximal cyclic (null) net in $G(\overline{R})$. For if not, by augmenting $\overline{M}'$ to form a maximal net $\overline{M}''$, the proper complement of $\overline{M}''$ would be a null (cyclic) net in $G(R)$ properly containing $M$, which is a contradiction. Moreover, if a maximal net $N$ is strongly connected to a maximal net $M$ in $G(R)$ where $N \neq M$, then $\overline{M}'$ is strongly connected to $\overline{N}'$ in $G(\overline{R})$. The maximal nets of $G(\overline{R})$ are thus complete and $G(\overline{R})$ also satisfies the condition of the Theorem. $G(R)$ and $G(\overline{R})$ are, therefore, permanent graphs.

*Necessity.* Assume $G(R)$ and its complement to be permanent graphs. We show that any two distinct maximal nets in $G(R)$ ($G(\overline{R})$) which are connected without an intervening maximal net between them, cannot be both universal nets or both articulated nets. There clearly must exist an intervening (maximal) universal net between any two distinct and connected articulated nets in $G(R)$ ($G(\overline{R})$). By virtue of the evident correspondence between maximal nets of $G(R)$ and $G(\overline{R})$, it then follows that an intervening articulated net must exist between any two distinct and connected (maximal) universal nets in $G(R)$ ($G(\overline{R})$). Finally, we show that any two distinct maximal nets in $G(R)$ ($G(\overline{R})$) are strongly connected. Assume $M$ and $N$ to be maximal nets, $M \neq N$, neither of which is connected to the other in $G(R)$ ($G(\overline{R})$). Then the proper complements $\overline{M}'$, $\overline{N}'$ are symmetrically connected universal nets in $G(\overline{R})$ ($G(R)$), which is a contradiction. Moreover, if $M$ is connected to $N$ in $G(R)$ ($G(\overline{R})$), $M$ must be strongly connected to $N$; for if not, $G(R)$ or $G(\overline{R})$ would not be complete, which is again a contradiction. Consequently, $G(R)$ and $G(\overline{R})$ are single chains of complete maximal nets which are alternately universal and articulated.

COROLLARY: *If* G(R) *and its complement are both permanent graphs,* G(R) *is symmetric if, and only if, it is the null graph or the universal graph.*

The following example provides an illustration of Theorem 3. Let $\rho$ denote the relation "less than, or equal to and odd" defined over the set of positive integers $S_n = \{k;\ 1 \leq k \leq n\}$; thus for $x, y \epsilon S_n$, $x \rho y$ if, and only if, $x < y$, or $x = y$ and is odd. The negation $\overline{\rho}$ of $\rho$ then denotes the relation "greater than, or equal to and even." The relations $\rho$, $\overline{\rho}$ are at once transitive and compact ($\rho \leq \rho^2$, $\overline{\rho} \leq \overline{\rho}^2$) so that the corresponding Boolean relation matrices $R, \overline{R}$ are both indempotent. $G(R)$ and $G(\overline{R})$ consequently consist of single chains of maximal nets of unit order which are alternately universal and articulated.

Not all idempotent relation matrices are accessible as limit matrices of nontrivial convergent sequences. An idempotent Boolean relation matrix $S$ is said to have a *proper power primitive* if, and only if, there exists a convergent relation matrix $R$ with limit matrix $R^* = S \neq R$ (cf. [6]).

THEOREM 4: *An idempotent Boolean relation matrix* S *of order* n ($n \geq 2$) *has no proper power primitive if, and only if,* G(S) *contains at least* $n-1$ *maximal universal nets, and any path between distinct vertices in* G(S) *contains exactly one acyclic edge.*

PROOF: *Sufficiency.* Assume the condition of the Theorem holds. Consider any convergent relation matrix $R$ with limit matrix $R^* = S$. We show $R = S$. The maximal nets of $G(R)$ and $G(S)$ coincide identically by Theorems 1 and 2. Any acyclic edge in $G(R)$ is in $G(S)$ for any such edge involves at least one vertex in a universal net which is by Theorem 1 in $G(S)$. Conversely, any acyclic edge in $G(S)$ is in $G(R)$. For if an edge $\overrightarrow{\alpha, \beta}$ of $G(S)$ is not in $G(R)$, there must exist an intervening path connecting $\alpha$ to $\beta$ in $G(R)$. Such a path contains at most one vertex $\gamma$ for all such vertices must be in null nets if only acyclic paths of unit length exist in $G(S)$. If $\alpha$, $\beta$ are both in univeral nets, then $\overrightarrow{\alpha, \gamma}$ and $\overrightarrow{\gamma, \beta}$ are in $G(S)$, which violates the hypothesis. If either $\alpha$ or $\beta$ is in a null net, with $\gamma$ there exist two articulated nets, which again violates the hypothesis. Consequently $\gamma$ cannot exist and the edge $\overrightarrow{\alpha, \beta}$ is in $G(R)$. $G(R)$ and $G(S)$ coincide and $R = S$.

*Necessity.* Assume $S$ is without proper power primitive. Consider the alternatives: (i) there exists a maximal net in $G(S)$ of order $m \geq 2$; (ii) $G(S)$ contains two or more articulated nets; (iii) there exists at least one path in $G(S)$ between distinct vertices involving two or more acyclic edges. If any of the three alternatives were to hold, it would be possible to construct a proper power primitive by adjoining (case (ii) and case (i) for null nets) or by removing (case (iii) and (i) for universal nets) a single edge in $G(S)$. The three alternatives are jointly excluded and lead to the condition of the Theorem. This completes the proof.

Theorem 4, in effect, states that an idempotent Boolean relation matrix $S$ of order $n$ ($n \geq 2$) has no proper power primitive if, and only if, $S$ contains an identity matrix of order $n-1$ and the intersection $S \cap \overline{I}_n$ is nilpotent, in fact $(S \cap \overline{I}_n)^2 = \Lambda_n$.

COROLLARY: *Every Boolean relation matrix* R *of order* $n \geq 3$ *with* G(R) *and* G($\overline{R}$) *both permanent graphs has a proper power primitive.*

PROOF: Directly by Theorems 3 and 4 (for $n=2$, the proposition fails).

We now define the graph of the sequence of powers of an arbitrary relation matrix and consider it briefly. The *power graph* $\Phi(R)$ of a Boolean relation matrix $R$ of order $n$ ($n \geq 1$) is given by the finite directed graph with labeled vertices corresponding to the distinct powers of $R$ and with edges existing only between successive powers. The *order* of $\Phi(R)$ is taken to be the cardinal number of vertices in $\Phi(R)$. $\Phi(R)$ clearly constitutes a many-one connected graph which is predominantly one-one with the exception of a distinguished pair of vertices which are both connected in one step to a unique vertex of $\Phi(R)$. The following properties of $\Phi(R)$ are readily verified: (i) $\Phi(R)$ is of order $m \leq (n-1)^2+1$; $m=(n-1)^2+1$ if, and only if, $G(R)$ is a universal net of order $n$($n \geq 2$) containing only two cycles, one of order $n$ and the other of order $n-1$ [9]. (ii) $\Phi(R)$ contains a single cycle which is terminal and of an order given by the period of $R$ (for $R$ convergent, the period is unity). (iii) Let $S$ denote the relation matrix corresponding to $\Phi(R)$; then the idempotent power of $S$ has no proper power primitive if, and only if, the order of $\Phi(R)$ exceeds the period of $R$ by at most unity (cf. Theorems 2 and 4). (iv) Consider the sequence of power graphs defined recursively by $\Phi^k(R) = \Phi(\Phi^{k-1}(R))$ for $k \geq 2$; then the order of $\Phi^w(R)$, where $w=(n-1)^2+1$, coincides with the period of $R$ (cf. (ii) above).

In the succeeding sections of the paper we treat two applications of the present results on idempotence.

## 4. Application in a Process Formulation of Consistent Orderings

As a simple application of some of the previous results, we briefly consider a stationary Boolean process formulation of rank orderings. In this approach, the particular properties of consistent order are taken to be absent (or not fully present) at the outset of the process, and are sequentially generated so as to be exhibited in complete form only at equilibrium term of the process. An ordering process or computation of this type might, for example, represent some aspects of the "process of weighing or balancing alternatives" in a problem of preference rankings (for a formally related approach to orderings, cf. [2]).

For present purposes, we select two types of consistent ordering, weak ordering and quasi-ordering. In a weak ordering $\rho$, the binary relation $\rho$ is required to be at once reflexive, transitive, and connected. Let $R$ denote the weak ordering relation matrix of order $n$ corresponding to $\rho$. $R$ then satisfies the following conditions: (i) $I_n \leq R$, (ii) $R^2 \leq R$, (iii) $\overline{I}_n \leq R \cup \overline{R}$. $R$ is clearly idempotent and the following identity obtains: (iv) $R \cup \overline{R} = V_n$. The identity states the complete "order-comparability" of all elements of the domain and range of $\rho$. In a quasi-ordering, comparability is not imposed as the binary relation satisfies only the two requirements of reflexivity and transitivity; the corresponding quasi-ordering matrix $R$ is again idempotent. We exclude the case of a strong ordering $\rho$ ($\rho$ at once irreflexive, connected, and transitive) in the present formulation for the obvious reason that any strong ordering matrix $R$ is nilpotent.

A finite sequence of successive powers of a Boolean relation matrix $R$ of order $n$ ($n \geq 2$) will be said to be a *weak ordering (quasi-ordering) process with generator or primitive* $R$ if, and only if, all powers are distinct, except for the ultimate and penultimate powers which coincide and are weak ordering (quasi-ordering) relation matrices. A weak ordering process or a quasi-ordering process is called *nontrivial* if the sequence contains at least three powers, and is otherwise termed *trivial*.

It is clear that examples of nontrivial weak ordering and quasi-ordering processes may be readily constructed. For any relation matrix $R$ such that $I_n \cup R \neq R_*$, the sequence $\{(I_n \cup R)^k;\ 1 \leq k \leq n\}$ contains a nontrivial quasi-ordering process in which the transitive closure $R_*$ is the terminal power. If in $G(R)$ of the preceding example, one at least of any two distinct vertices is accessible from the other (cf. (iv) above), then the sequence contains a nontrivial weak ordering process.

We have the following result on ordering processes.

PROPOSITION: *For any weak ordering relation matrix* R *of order* $n \geq 3$, *there exists a nontrivial weak ordering process with* R *as terminal power.*

PROOF: The result follows from Theorem 4 by showing that $R$ must have a proper power primitive (for $n=2$, the proposition fails). Assume the weak ordering matrix $R$ has no proper power primitive. Then $G(R)$ contains only cycles of unit order. Consider three distinct vertices $\alpha$, $\beta$, $\gamma$ and suppose $G(R)$ to contain an edge $\overrightarrow{\alpha,\beta}$. Then one of the following triads must exist in $G(R)$: (i) $\overrightarrow{\alpha,\beta}$, $\overrightarrow{\beta,\gamma}$, $\overrightarrow{\alpha,\gamma}$; (ii) $\overrightarrow{\alpha,\beta}$, $\overrightarrow{\gamma,\beta}$, $\overrightarrow{\gamma,\alpha}$; (iii) $\overrightarrow{\alpha,\beta}$, $\overrightarrow{\gamma,\beta}$, $\overrightarrow{\alpha,\gamma}$. The graph then exhibits a path involving two acyclic edges, which is a contradiction. $R$ therefore has a proper power primitive and is the limit matrix of some nontrivial weak ordering process.

The earlier results (and the observations on power graphs) consequently provide a basis for the design of processes of varying length for the evolution of consistent orderings where the latter, in effect, arise in the attainment of "maximal connectivities." Although we do not consider these further here, ordering processes of a more general character may be introduced as "subrelations" of terms in sequences which converge to limit forms of non-ordering type; as may, for example, be carried through with the aid of Theorem 3 (cf. Corollary, Theorem 4). Moreover, maximal null nets could in ranking problems be formally treated as representations of incommensurability.

## 5. Application in an Information System Model

We examine, lastly, the structure of a recursive model of a large-scale information system which was first proposed and studied in connection with a particular logical formulation of processes of communication or diffusion of information [8]. The model was originally stated in the formalism of the calculus of relations, but some of its basic properties were investigated, with greater convenience, by graph-theoretic means and associated relation matrix methods. In the earlier study, principal interest attached to the structure of transient information flow, and to algorithms for the computation of "minimum transmission times" in the propagation of information. At this point, we fix attention on issues of equilibrium or steady-state information flow. This leads naturally to an abstract connectivity approach to issues relating to the accessibility of information. In the following, we adhere for the most part to the order of development and terminology of the earlier study; some alterations of notation and language have, however, been made in the interests of clarity and conciseness [8, 10].

We consider a recursive system $\Omega(k)$ defined for all nonnegative integers $k$, as a composition of three fundamental binary relations, of which two are homogeneous and one nonhomogeneous. Two arbitrary and distinct sets $\Delta$, $\Sigma$ are assumed to be given, each finite and non-empty. A homogeneous binary relation $\rho$ called the *communication relation* is defined on the elements of $\Delta$. A homogeneous binary relation $\sigma$ called the *derivation relation* (and also called the *primitive implication relation*) is defined on the elements of $\Sigma$; $\sigma$ is not postulated to be either reflexive or transitive. Finally, a nonhomogeneous binary relation $\tau$ called the *assignment relation* is defined on the product set $\Delta \times \Sigma$, with domain and range respectively contained in $\Delta$ and $\Sigma$.

An element of the field (i.e., the domain or the range) of $\rho$ is called a *communication element or entity*, and $x\rho y$ is read "$x$ communicates with $y$." An element of the field of $\sigma$ is called an *information element or entity*, where such entities may be of arbitrary logical or mathematical character; $\alpha\sigma\beta$ is read "$\beta$ is derived from $\alpha$" or "$\alpha$ primitively implies $\beta$." The relation $\tau$ is regarded as an "initial input relation" in the recursive system; for $x\epsilon\Delta, \alpha\epsilon\Sigma, x\tau\alpha$ is read "$x$ is assigned and holds $\alpha$."

The model structure is based on the perspicuous and well-known principle of computation in certain finite and closed information systems to the effect that *information is produced only by the operations of (initial) assignment, communication, and derivation.* In the present structure, communication and derivation are treated as equivalent or substitutable operations in respect to transmission of information and, in fact, proceed at the same unit rate. The recursive system of interest is completely defined by

$$\Omega(0) = \tau, \tag{1a}$$

$$\Omega(n) = \breve{\rho}\Omega(n-1) \cup \Omega(n-1)\sigma, \quad (n = 1, 2, \ldots), \tag{1b}$$

where $\breve{\rho}$ is the converse of $\rho$. $\Omega(n)$ is thus given as the union of the right and left relative products of $\Omega(n-1)$ by $\sigma$ and $\breve{\rho}$ respectively, and with domain and range corresponding to that of $\tau$. The relation $\Omega(k)$ is called the *thesaurus relation of order* $k$ ($k \geq 0$); for any integer $k$, $x\epsilon\Delta, \alpha\epsilon\Sigma$, $x\Omega(k)\alpha$ is read "$x$ holds $\alpha$ at stage $k$." Thus, (1b) may be regarded as providing a system analogue of *modus ponens*[2] in the form: If $x\Omega(k)\alpha$ and $\alpha\sigma\beta$, then $x\Omega(k+1)\beta$ ($x\epsilon\Delta; \alpha, \beta\epsilon\Sigma$). In homologous fashion, the basic triadic relation "$x$ communicates $\alpha$ to $y$" is rendered in the form: If $x\Omega(k)\alpha$ and $x\rho y$, then $y\Omega(k+1)\alpha$ ($x, y\epsilon\Delta; \alpha\epsilon\Sigma$). The relations $\rho$ and $\sigma$ may then be regarded as expedient means of treating the fundamental operations of transfer and logical detachment. In a recent study, the relation $\tau$ of the present model has been construed as a state-description (in relation matrix form) of an information store.[3] The recursive system, in effect, affords a composite representation of two processes for the propagation of information which abstracts virtually every aspect except that of connectivity.

We turn to the problem of convergence in the system defined by (1a) and (1b). For this purpose, a sequence $\{\Omega(k); k \leq 0\}$ will be said to be *convergent* if, and only if, there exists a term $\Omega(p)$ in the sequence such that $\Omega(p) = \Omega(p+1)$. Let $\Omega_*$ denote the union $\bigcup_{k=0}^{\infty} \Omega(k)$ of all terms in a sequence (convergent or not), and let $\Omega^*$ denote the unique limit term of a convergent sequence. $\Omega_*$ and $\Omega^*$ are respectively called the *thesaurus relation closure* and the *thesaurus relation limit*.

From (1a) and (1b) we obtain in the powers of $\breve{\rho}$ and $\sigma$

$$\Omega(m) = \bigcup_{h=0}^{m} \breve{\rho}^h \tau \sigma^{m-h} \quad (m = 0, 1, 2, \ldots), \tag{2}$$

where $\breve{\rho}^0$, $\sigma^0$ are identity relations respectively defined on $\Delta$ and $\Sigma$. The system (1a), (1b) exhibits redundancy for $m \geq 2$ since

$$\begin{aligned} \Omega(m) &= \breve{\rho}\Omega(m-1) \cup \tau\sigma^m \\ &= \breve{\rho}^m\tau \cup \Omega(m-1)\sigma, \end{aligned}$$

and $(\breve{\rho}\Omega(m-1)) \cap (\Omega(m-1)\sigma) \geq \bigcup_{h=1}^{m-1} \breve{\rho}^h \tau \sigma^{m-h}$.

If both $\sigma$ and $\rho$ were at once reflexive and transitive, the thesaurus relation sequence $\{\Omega(k); k \geq 0\}$ would converge rapidly. For then by (2), $\Omega^* = \Omega(k) = \breve{\rho}\tau \cup \tau\sigma \cup \breve{\rho}\tau\sigma$ for all $k \geq 2$. Moreover, $\Omega_* = \tau \cup \Omega^*$.

The general problem of convergence may be treated with advantage by the device of embedding the original system of recursions in a larger recursive system of

[2] Cf. Paul Hertz "Über Axiomensysteme für beliebige Satzsysteme," Math. Ann. 87 (1922), pp. 246–269.

[3] D. B. Hertz et al., "Research Study of Criteria and Procedures for Evaluating Scientific Information Retrieval Systems," Final Report (Contract NSF C-218), March 1962, prepared for Office of Scientific Information Service, National Science Foundation.

known structure. We proceed to reformulate the original recursive system as a stationary Boolean process (discrete parameter). Let $R$, $S$, $T$ be Boolean relation matrices corresponding to $\rho$, $\sigma$, $\tau$ and let the sequence of rectangular Boolean matrices $\{W(k); k \geq 0\}$ (called the sequence of *information configurations*) correspond to the sequence $\{\Omega(k); k \geq 0\}$. Moreover, let $W^*$, $W_*$ respectively correspond to $\Omega^*$ and $\Omega$ . The original system is then restated as

$$W(0) = T, \tag{1a'}$$

$$W(n) = \breve{R}W(n-1) \cup W(n-1)S, \qquad (n = 1, 2, \ldots), \tag{1b'}$$

and the explicit form of $W(m)$ is given by

$$W(m) = \bigcup_{h=0}^{m} \breve{R}^h TS^{m-h} \qquad (m = 0, 1, 2, \ldots). \tag{2'}$$

The last system of recursions can be concisely reformulated. Let $Q$ denote the square Boolean relation matrix $\begin{bmatrix} \breve{R} & T \\ \varnothing & S \end{bmatrix}$, where $\varnothing$ denotes a rectangular null matrix. The new recursion constitutes a stationary Boolean process in the powers of $Q$ and yields

$$Q^n = Q^{n-1}Q = \begin{bmatrix} \breve{R}^n & W(n-1) \\ \varnothing & S^n \end{bmatrix} \qquad (n = 1, 2, \ldots), \tag{3}$$

where in nonredundant manner $W(n) = \breve{R}W(n-1) \cup TS^n = \breve{R}^n T \cup W(n-1)S$. The convergent behavior of the sequence $\{W(k); k \geq 0\}$ is then simply subsumed in the asymptotic behavior of the powers of $Q$. By Lemma 2c it is clear that a $W(k)$-sequence may converge even though $R$ or $S$ or both are oscillatory in their powers. For example, let the graph $G(Q)$ be specified as follows: (i) $G(\breve{R})$, $G(S)$ are respectively cycles of order $r$ and $s$ such that $(r, s) = 1$; (ii) $G(T)$ contains only one edge. The $W(k)$-sequence then converges, and $W(rs-1)$ in $Q^{rs}$ consists exclusively of nonnull entries.

More generally, Lemma 2c provides an essentially complete graph-theoretic characterization of the convergence conditions for $W(k)$-sequences and more, in the present embedding. In fact, for $x \epsilon \Delta$, $\eta \epsilon \Sigma$, $x\Omega^*\eta$ if, and only if, for some positive integer $q_0$ there exist one or more cyclic traverses from vertex $\alpha_x$ to vertex $\alpha_\eta$ in $G(Q)$ (involving communication relation or derivation relation cyclic traverses) which exhibit paths of all integral lengths $q \geq q_0$. It is worth noting that a single maximal null net of $G(\breve{R})$ may contribute to two or more maximal null nets in $G(Q)$ depending on the structure of the "coupling subgraph" $G(T)$. From (3) we readily conclude by Lemma 1 and Theorem 1 that a $W(k)$-sequence converges if $R$ and $S$ are both convergent. Consequently, if $\rho$, $\sigma$ are both drawn from one of the following *classes of relations* the $W(k)$-sequence and the $\Omega(k)$-sequence exhibit convergent behavior (cf. [9]): reflexive, transitive, compact, equivalence, weak ordering, quasi-ordering, serial (strong ordering), hierarchical. From the preceding remark it also follows that the transitive closure $Q_*$ of $Q$ contains the union $W_*$, whether the $W(k)$-sequence is convergent or not.

We turn to a derivation of the explicit forms of $W^*$ and $W_*$ in the general case. Let $\breve{R}$, $S$ be of dimensions $d_r$, $d_s$. Let $\Gamma(\breve{R})$, $\Gamma(S)$ denote the groups of limit matrices of $\breve{R}$ and $S$ and of periods $p_r$, $p_s$. Let $\breve{R}^{**}$, $S^{**}$ denote the unions of the elements of $\Gamma(\breve{R})$ and $\Gamma(S)$. Let $\Gamma(Q)$ denote the group of limit matrices of $Q$ of period $p_q$, where $p_q \leq$ l.c.m. $(p_r, p_s)$. An element $Q'_\alpha$ of $\Gamma(Q)$ will be written in the form

$$Q'_\alpha = \begin{bmatrix} \breve{R}'_\alpha & W_\alpha \\ \varnothing & S'_\alpha \end{bmatrix} \qquad (\alpha = 1, \ldots, p_q),$$

where $\breve{R}'_\alpha \epsilon \Gamma(\breve{R})$, $S'_\alpha \epsilon \Gamma(S)$. Let $Q^{**}$ denote the union $\bigcup_{\alpha=1}^{p_q} Q'_\alpha$ and let $Q'_1$ denote the identity of $\Gamma(Q)$: $\breve{R}'_1$, $S'_1$ are then the identities of $\Gamma(\breve{R})$, $\Gamma(S)$.

The group identity $Q'_1$ is idempotent so that

$$W_1 = \breve{R}'_1 W_1 \cup W_1 S'_1. \tag{4}$$

Assume now that the $W(k)$-sequence converges. The idempotent matrix $Q^{**}$ then leads to the relation

$$W^* = \breve{R}^{**}W^* \cup W^*S^{**}. \tag{5}$$

From the equations $QQ^{**} = Q^{**}Q = Q^{**}$ we obtain

$$W^* = \breve{R}W^* \cup TS^{**}, \tag{6a}$$

$$W^* = W^*S \cup \breve{R}^{**}T. \tag{6b}$$

The relations $\breve{R}^k TS^{**} \leq W^*$, $\breve{R}^{**}TS^k \leq W^*$ for all $k \geq 0$ follow from (6a) and (6b) respectively. The inclusion $\breve{R}^{**}TS \cup \breve{R}_*TS^{**} \leq W^*$ then follows readily ($R_*$, $S_*$ are the transitive closures) and the converse inclusion follows directly from (2') and (5) so that

$$W^* = \breve{R}^{**}TS_* \cup \breve{R}_*TS^{**}. \tag{7}$$

If both $\breve{R}$ and $S$ are convergent, we write $\breve{R}^*$ for $\breve{R}^{**} = \breve{R}'_1$ and $S^*$ for $S^{**} = S'_1$ so that (7) becomes

$$W^* = \breve{R}^*TS_* \cup \breve{R}_*TS^*. \tag{8}$$

The expression for $W_*$ follows directly from (8) and is given by

$$W_* = \breve{R}_*TS_*. \tag{9}$$

From (4) it is clear that the $W(k)$-sequence converges if, and only if, $W_1 = \breve{R}'_1 W_1 \cup W_1 S'_1 = \breve{R}^{**}TS_* \cup \breve{R}_*TS^{**}$. This condition may be seen to imply the following two conditions for the group unions $\breve{R}^{**}$, $S^{**}$:

$$\breve{R}^{**}T \leq \breve{R}'_1 T \cup \breve{R}^{**}TS, \tag{10a}$$

$$TS^{**} \leq TS'_1 \cup \breve{R}TS^{**}, \tag{10b}$$

the condition $W_\alpha = W^*$ can be shown to be equivalent to each of the following:

$$W_\alpha = \breve{R}'_\alpha TS_\cdot \cup \breve{R}_* TS^{**}, \tag{11a}$$

$$W_\alpha = \breve{R}_\cdot TS'_\alpha \cup \breve{R}^{**} TS_*, \quad (\alpha = 1, \ldots, p_Q) \tag{11b}$$

Finally, for convergent $W(k)$-sequences it may be verified that $[\max(d_s, d_r) - 1]^2 + \min(d_s, d_r)$ provides a sharp upper bound for the smallest integer $q$ such that $W(q) = W^*$ (cf. [9]).

The expressions for $W^*$ and $W_*$ in (8) and (9) are of particular interest in the light of the present results on idempotence. The graphs $G(W^*)$ and $G(W_*)$ are by the embedding represented as subgraphs respectively of the permanent graphs $G(Q^{**})$ and $G(Q_*)$ of the composite information process. Theorems 1 and 2 (and Lemmas 1 and 2c as well) consequently provide the basis for a detailed description of the thesaurus relation limit $\Omega^*$ and the thesaurus relation closure $\Omega_*$. The graph $G(Q^{**})$ could satisfy the requirements of Theorem 3. The formulas for $W^*$ and $W_*$ algebraically exhibit the components of information accessibility in the form of closure expressions of subsystem connectivity in $\breve{R}_*$ and $S_*$. By virtue of the presence of $\breve{R}^{**}$, $S^{**}$ in the expression for $W^*$ it is quite evident that the maximal nets of $G(R)$ and $G(S)$ constitute fundamental units of analysis in the study of the recursive system. This in turn leads to the clear possibility of constructing equivalence classes of communication elements relative to specified maximal nets of the graph of the derivation relation which may be called "information-accessibility equivalence classes." Such constructions could be effected for $W_*$ as well as $W^*$.

The matrix $W^*$ of a convergent sequence (for given $R$, $T$, and $S$) is viewed as the equilibrium or steady-state information configuration of the recursive system. For many significant information sequences, $W^*$ is identically null since $R$ and $S$ are both null convergent, e.g., $R$, $S$ both strict hierarchies with graphs $G(R)$, $G(S)$ given as trees. In such cases, the closure $W_*$ regarded as the summary computational history of the sequence provides a useful description of the information process. In systems of greater complexity $W^*$ is not identically null (or the limit does not indeed exist) so that the persistence of nonnull information configurations is always mediated through "reverberating circuits" in the form of maximal cyclic nets of $G(R)$ or of $G(S)$.

The organization of information in the present recursive system may be outlined with the aid of (8) and (9), and a supplementary classification of maximal nets. Theorem 4 and the earlier noted "information-accessibility equivalence classes" for $W^*$ and $W_*$ are also relevant here. We consider a "hierarchical" classification of the maximal nets of $G(R)$ and $G(S)$. A maximal net (null or cyclic) of $G(R)$ ($G(S)$) is said to be of the *first level* if no vertex of the net is attainable from any other maximal net of $G(R)$ ($G(S)$). A maximal net of $G(R)$ ($G(S)$) is recursively said to be of *level* $k$ if, and only if, it is attainable in one step from a distinct maximal net of level $k-1$ and is not attainable in one step from any maximal net of lower level. In the general case, $G(R)$ and $G(S)$ are each possibly composed of disjoint and frequently similar (i.e., one-one corresponding) subgraphs. The transmission and distribution of information in the present recursive system for specified $R$ and $S$ are then governed by the mode of assignment of the maximal net "hierarchies" of $G(R)$ to the maximal net "hierarchies" of $G(S)$. It would, consequently, be of interest to consider alternative general system designs for the achievement of preassigned $W^*$ or $W_*$.

---

This work was supported in part by the U.S. Patent Office.

## 6. References


(1) F. E. Hohn and L. R. Schissler, Boolean matrices and the design of combinational relay switching circuits, Bell Syst. Tech. J. **34** 177–202, (1955).

(2) M. G. Kendall, Further contributions to the theory of paired comparisons, Biometrics **11** 43–62, (1955).

(3) A. G. Lunts, The application of Boolean matrix algebra to the analysis and synthesis of relay contact networks, Dokl. *Akad. Nauk SSSR* **70** 421–423, (1950). (In Russian).

(4) O. Ore, Theory of Graphs, Am. Math. Soc., Providence (1962).

(5) C. S. Peirce, Description of a notation for the logic of relatives, resulting from an amplification of the conceptions of Boole's calculus of logic, Memoirs of the American Academy **9** 317–378, (1870) reprinted in Collected Papers **III**, 27–98 (Exact Logic) (Harvard Univ. Press, Cambridge, 1933).

(6) C. S. Peirce, Nomenclature and divisions of dyadic relations, (1903), reprinted in Collected Papers **III**, 366–387 (Exact Logic) (Harvard Univ. Press, Cambridge, 1933).

(7) R. T. Prosser, Applications of Boolean matrices to the analysis of flow diagrams, Tech. Rept. No. 217, Massachusetts Institute of Technology, Lincoln Laboratory, Lexington, (1960).

(8) D. Rosenblatt, A note on communication in organizations, Carnegie Institute of Technology, Pittsburgh, (unpublished) 1951.

(9) D. Rosenblatt, On the graphs and asymptotic forms of finite Boolean relation matrices and stochastic matrices, Naval Research Logistics Quarterly **4**, 151–167, (1957).

(10) D. Rosenblatt, On some aspects of models of complex behavioral systems, in Information and Decision Processes, pp. 62–86 (R. E. Machol, ed.), (McGraw-Hill Book Co., New York, N.Y., 1960).


(Paper 67B4–107)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 67B, No. 4, October–December 1963

# Publications of the National Bureau of Standards*

## Selected Abstracts

**The maximum number of zeros in the powers of an undecomposable matrix,** M. Marcus and F. May, *Duke Math. J.* **20,** *No. 4, 581–588 (Dec. 1962).*
Let $A$ be an $N$-square matrix. $A$ is called decomposable if there is a permutation matrix $P$ such that $PAP^T$ is a subdirect sum. The question here is: given an $n$-square indecomposable matrix with complex entries, how many fixed positions $(i, j)$, $1 \leq i, j \leq n$ can be zero in every positive integral power of $A$. This is answered for several classes of matrices, including normal matrices with distinct characteristic roots.

**Packing inequalities for circles,** P. J. Davis, *Michigan Math. J.* **10,** *25–31 (1963).*
The theory of complex variables is employed to obtain inequalities for non overlapping circles.

**Lower bounds to eigenvalues using operator decompositions of the form B*B,** N. W. Bazley and D. W. Fox, *Arch. Rat. Mech. Anal.* **10,** *352 (1962).*
A procedure is given for obtaining lower bounds to the eigenvalues of a self-adjoint operator; the difference between the quadratic form of the operator and that of a second operator having a smaller quadratic form is assumed to be of the form $(Bu, Bu)'$ in which $B$ is an operator from one Hilbert space $\mathfrak{H}$ to another $\mathfrak{H}'$. The eigenvalues and eigenvectors of the smaller quadratic form are also assumed known. The lower bounds are obtained by instructing operators that have quadratic forms intermediate between the smaller and that of the given operator. The construction depends on the use of the operator $B^*$ adjoint to $B$.

**Statistical interpretations,** W. J. Youden, *Book, Standard Methods of Chemical Analysis* **IIA,** *Chapt. 16, 319–325 (D. Van Nostrand Co., Princeton, New Jersey, N.J. 1963).*
This paper has been prepared for the chapter on Statistics in the new (6th) edition of Scott's Standard Methods of Chemical Analysis published by Van Nostrand. The first part of the paper goes into detail regarding the steps required to obtain meaningful estimates of the errors in analytical determinations. The remainder takes up a few recurring problems in analytical chemistry along with the appropriate statistical techniques.

**Precision of simultaneous measurement procedures,** W. A. Thompson, Jr., *J. Am. Stat. Assoc.* **58,** *No. 302, 474–479 (June 1963).*
We consider the problem of measurement under the following conditions: The process of gathering the data is such that on any given item only one opportunity for measurement occurs, but it can be observed simultaneously by several instruments. The items to be measured are variable so that one cannot obtain replicate observations with the same instrument which would show directly the variance of the instrument readings. Procedures are discussed for estimating the precisions of the instruments and the variability of the items being measured.

**Determining fastest routes using fixed schedules,** B. M. Levin and S. Hedetniemi, *Proc. Conf. American Federation for Information Processing Soc.* **23,** *1–9 (1963).*
In determining routings between locations having no direct connections, most published timetables leave to the patron the job of selecting the most desirable routing. A program has been written which determines routings using speed and cost as criteria. The solution stems from and is related to the Shortest Route Problem. This paper discusses the program and its relationship to the Shortest Route Problem.
The present program is designed for use in routing airmail. The techniques can be used to route passengers, or articles on any type of scheduled transportation trip. It can also be useful in evaluating proposed transportation networks.

**Dielectric relaxation in a high temperature dipole lattice,** R. Zwanzig, *J. Chem. Phys.* **38,** *No. 11, 2766 (1963).*
The effect of rotational Brownian motion on the dielectric susceptibility of a rigid cubic lattice of permanent point dipoles is calculated. All dipolar interactions are taken into account by means of a high temperature perturbation expansion. The most significant result is that dipolar interactions give rise to additional new relaxation times, increasing dielectric loss at high frequencies.

**Some free products of cyclic groups,** M. Newman, *Michigan Math. J.* **9,** *369–373 (1962).*
Let $\lambda_n = 2\cos(\pi/n)$. The principal result of this paper is that the group generated by the linear fractional transformations

$$\tau' = -1/(\tau - \lambda_p),\ \tau' = -1/(\tau + \lambda_q)$$

is the free product of a cyclic group of order $p$ and a cyclic group of order $q$ for $p \geq 2$, $q \geq 3$.

**Multipliers of difference sets,** M. Newman, *Can. J. Math. 121–124 (Nov. 1961).*
Let $D$ be a difference set with parameters $v, k, \lambda, v > k > \lambda > 0$. Let $q$ be a prime divisor of $n = k - \lambda$. The Hall-Ryser theorem states that if $(q, v) = 1$ and $q > \lambda$ then $q$ is a multiplier of $D$. In this paper a proof of theorem is given by incidence matrices alone. The restriction $q > \lambda$ is also removed in certain cases, e.g. $n = q$, or $n = 2q$ and $(v, 7) = 1$.

**The segemental variation of Blaschke products,** G. T. Cargo, *Duke Math. J.* **30,** *No. 1, 143–150 (Mar. 1963).*
We say that a function which is regular in the open unit disk $D$ has finite segmental variation at a point $e^{i\theta}$ provided every line segment connecting $e^{i\theta}$ to a point of $D$ is mapped onto a rectifiable curve by the function.
Seidel and Walsh proved that a univalent function has finite segmental variation almost everywhere; and Tsuji proved that, if a function has a finite Dirichlet integral, it has finite segmental variation off a set whose (logarithmic) capacity is zero.
In this paper, analogous local and global theorems are established for a well-known class of functions whose members are not univalent and do not have finite Dirichlet integrals, namely, (infinite) Blaschke products.

**The structure of some subgroups of the modular group,** M. Newman, *Illinois J. Math.* **6,** *No. 3, 480–487 (Sept. 1962).*
The groups $\Gamma^m$ generated by the $m$th powers of all elements of the 2×2 modular group $\Gamma$ are studied and their structure determined. The connection with the Burnside problem is discussed.

**Conditions for second-order waves in hypo-elasticity,** B. Bernstein, *Trans. Soc. Rheology* **VI,** *No. 1, 263–273 (1962).*
The study in hypo-elasticity of second order waves, regarded as propagating singular surfaces of order two in displacement, yields results similar to those in general elasticity theory. The amplitude of a wave must be an eigenvector of a certain matrix, the acoustical tensor, which depends on the stress and direction of propagation, and the product square of the wave speed with the mass density must be the corresponding eigenvalue. Conditions are stated that at least one wave is possible in each direction and that the existence of a longitudinal amplitude imply the possibility of a longitudinal wave. The condition that the acoustical tensor be always symmetric is the same as that a Cauchy-elastic hypo-elastic material possess a strain energy.

**The invariance of symmetric functions of singular values,** M. Marcus and H. Minc, *Pacific J. Math.* **12,** *No. 1, 327–332 (1962).*

Let $M$ denote the vector space of all $m \times n$ matrices over the complex numbers and let $T$ be a linear map of $M$ into itself. Let $f(A)$ denote the $r^{th}$ elementary symmetric function of the squares of the singular values of $A$, $A \in M$. We determine completely the structure of the group $M_f$ of all linear $T$ satisfying the invariance condition: $f(T(A)) = f(A)$ for all $A \in M$.

**Bounds for cofactors and arithmetic minima of quadratic forms,** M. Newman, *J. London Math. Soc.* **38,** *p. 215–217 (Apr. 1963).*
Let $A$ be a real, symmetric $n \times n$ positive definite matrix of determinant $d$. It is shown that after a suitable integral congruence transformation bounds for the determinants of the leading principal minor matrices of $A$ and for the diagonal elements of $A$ may be derived which depend only on $n$ and $d$. It is also shown that the Hermite constant $\gamma_n$ satisfies

$$\gamma_{m+n}^{m+n} \geq \gamma_m^m \gamma_n^n.$$

**Convergence to normality of powers of a normal random variable,** N. Severo and L. J. Montzingo, Jr., *Bull. Intern. Stat. Inst.* **XXXIX,** *Pt. 2, 491–500 (1962).*
Let $Y$ be a normally distributed random variable, and let $p$ be a real number such that $Y^p$ is also real, $-\infty \leq y \leq +\infty$. Let $\mu_y$ and $\sigma_y$ be the mean and standard deviation of $Y$, respectively. Let $X = Y^p$. It is shown that if $p$ is a positive integer, then $X$ is asymptotically normally distributed with mean $\mu_x$ and standard deviation $\sigma_{x'}$ as $\mu_y/\sigma_y \to \infty$. Otherwise, $X$ is asymptotically normally distributed with mean $\mu_0 = \mu_y^p[1 + 0(\eta^{-2})]$ and standard deviation

$$\sigma_0 = (|p|\mu_y^{p-1})[1 + 0(\eta^{-2})]^{1/2}$$

as $\mu_y/\sigma_y \to \infty$. When $\mu_x$ and $\sigma_x$ exist, it is shown that $\mu_y/\sigma_y \to \infty$ is necessary and sufficient to insure that $\mu_x/\sigma_x \to \infty$.

**Non-linear effects in spectra of the iron group,** R. E. Trees, *Phys. Rev.* **129,** *No. 3, 1220–1224 (Feb. 1, 1963).*
As known for the last ten years, calculations made with the linear theory agree poorly with experiment in the $3d^6$ configuration of Fe III, compared to the agreement obtained for the $3d^64s$ configuration of that spectrum. It is shown that effects of $3s3d^64s$ on $3s^23d^64s$ satisfy the necessary conditions for linear behavior, consistent with the close agreement obtained in $3d^64s$ of Fe III. Effects of $3s3d^7$ on $3s^23d^6$ are not linear; it is shown that if they are accounted for in the calculations, then close agreement is obtained in the $3d^6$ configuration also (the mean error is $\pm 65$ cm$^{-1}$). The magnitude of the parameter $\gamma$, introduced to define this non-linear interaction, is found to be 40% smaller than expected from Hartree-Fock calculations, but this is a minor inconsistency in view of the known inadequacies of s.c.f. orbitals. It is pointed out that the parameters of the linear theory are changed considerably when the non-linear effects are accounted for.

**Normal functions, the Montel property, and interpolation in $H^\infty$,** G. T. Cargo, *Michigan Math. J.* **10,** *141–146 (1963).*
This paper exhibits an interrelationship among three apparently unrelated subjects, namely, normal functions, Montel's property, and interpolation with bounded holomorphic functions in the open unit disk. As a by-product, new and somewhat shorter proofs of several known results are obtained.

**Copositive and completely positive quadratic forms,** *M. Hall,* Jr., and M. Newman, *Proc. Camb. Phil. Soc.* **59,** *329–339 (1963).*
The relation between quadratic forms which are sums of squares of non-negative linear forms, and quadratic forms which are non-negative for non-negative values of the variables is investigated and a duality theorem is proved.

**Covers and packings in a family of sets,** J. Edmonds, *Bull. Am. Math. Soc.* **LXVIII,** *No. 5, 494–499 (Sept. 1962).*
A graph-theoretic characterization is given of (1) minimum-cardinality covers of a set $S$ chosen from a specified family $F$ of its subsets and (2) maximum-cardinality families of mutually disjoint sets chosen from $F$.

**On the pedestrian queueing problem,** G. Weiss, *Bull. Intern. Stat. Inst.* **XXXIX,** *No. 4, 163–168 (1962).*
J. C. Tanner has discussed the problem of the delay to pedestrians trying to cross a road, and being blocked by traffic. In his treatment it was assumed that the pedestrians arrived at the intersection completely at random and crossed either upon arrival or in a group, after the departure of a car. Tanner also assumed that the criterion for crossing depended on whether the gap seen by the pedestrian exceeded a given one, $T_c$, and that the gaps were described by a Poisson distribution. Mayne generalized this problem by allowing arbitrary gap and arrival distributions with the gap acceptance criterion given by a critical gap $T_c$, as in Tanner's treatment. We shall consider the same problem assuming a Poisson arrival distribution for pedestrians, but allowing for a probabilistic gap acceptance criterion, i.e., we let $\alpha(t)$ be the probability that a pedestrian will cross the road if he sees a gap of $t$ in the oncoming traffic. In this paper the imbedded Markoff process generated by this situation will be studied, an expression for the generating function for the number of pedestrians at the intersection will be given, and various other statistics discussed.

**A connection between Tauberian theorems and normal functions,** G. T. Cargo, *Bull. Am. Math. Soc.* **68,** *No. 4, 400–401 (July 1962).*
In this note it is pointed out that certain Tauberian theorems follow immediately from some recent research of Lehto-Virtanen and Bagemihl-Seidel.

**Note on a subgroup of the modular group,** M. Newman and J. R. Smart, *Proc. Am. Math. Soc.* **14,** *No. 1, 102–104 (Feb. 1963).*
Let $\Gamma$ be the $2 \times 2$ modular group, $\Gamma^2$ the subgroup of $\Gamma$ generated by the squares of the elements of $\Gamma$, $\Gamma^3$ the subgroup of $\Gamma$ generated by the cubes of the elements of $\Gamma$. Let $G'$ denote the commutator subgroup of $G$ for any subgroup $G$ of $\Gamma$. Then it is shown that

$$(\Gamma^2)'(\Gamma^3)' = \Gamma',$$
$$(\Gamma^2)' \cap (\Gamma^3)' = \Gamma(6)$$

where $\Gamma(6)$ is the principal congruence subgroup of $\Gamma$ of level 6.

**Linear velocity-gradient term in time-dependent pair distribution function,** R. E. Nettleton, *J. Chem. Phys.* **38,** *No. 8, 179–1802 (Apr. 1963).*
A Chapman-Enskog type perturbation solution, in gradients of energy, velocity, and density, is proposed for the hierarchy of integro-differential equations obeyed by the time-dependent Ursell-Mayer functions. The hierarchy is then terminated by an *ansatz* relating the three-particle Ursell functions to those of lower order, which yields a closed system for the perturbations to the singlet and pair distribution functions (it is assumed the equilibrium functions are known). Attention is focused on the equation for the two-particle functions, and on the term therein proportional to the divergence of fluid velocity, from which the bulk viscosity may be calculated. An expansion for this term is assumed in powers of scalar products of particle separations and velocities, in which the coefficients in turn are expanded in Sonine polynomials in the velocities. The functions which multiply these polynomials satisfy a system of integral equations, for which a first approximation is written down and the solution discussed. Finally, it is shown that substitution of the solution for the pair distribution function into the hierarchy equation for the singlet distribution yields an infinite number of conditions which may be used to evaluate a corresponding number of parameters in the *ansatz* employed to close the hierarchy.

## Other NBS Publications

**J. Res. NBS 67A (Phys. and Chem.), No. 5 (Sept.–Oct. 1963) 70 cents.**

Reduction of space groups to subgroups by homogeneous strain. H. S. Peiser, J. B. Wachtman, Jr., and R. W. Dickson.

High-temperature thermodynamic functions for zirconium and unsaturated zirconium hydrides. T. B. Douglas.

Heat of oxidation of aqueous sulfur dioxide with gaseous chlorine. W. H. Johnson and J. R. Ambrose.

Thickness of adsorbed polystyrene layers by ellipsometry. R. R. Stromberg, E. Passaglia, and D. J. Tutas.

Melting temperature and change of lamellar thickness with time for bulk polyethylene. J. J. Weeks.

Precise coulometric titrations of potassium dichromate. G. Marinenko and J. K. Taylor.

Resolution of limits of analyzers and oscillatory systems. E. L. R. Corliss.
Synthesis, purification, and physical properties of seven twelve-carbon hydrocarbons. T. W. Mears, C. L. Stanley, E. L. Compere, Jr., and F. L. Howard.
Reactions of polyfluorobenzenes with nucleophilic reagents. L. A. Wall, W. J. Pummer, J. E. Fearn, and J. M. Antonucci.

**J. Res. NBS 67C (Eng. and Instr.), No. 4 (Oct.–Dec. 1963) 75 cents.**

Limitations on electron beam density of unipotential electron guns at low voltages. J. A. Simpson and C. E. Kuyatt.
New radiofrequency mass spectrometer having high duty cycle. R. H. Mills.
A power-series buildup factor formulation. Application to rectangular and off-axis disk source problems. J. B. Hubbell.
An alinement interferometer. J. B. Saunders.
Determination of optical path difference for a photographic objective. F. E. Washer and W. R. Darling.
Development of filters for a thermoelectric colorimeter. S. H. Emara and R. P. Teele.
Application of air bearings to an electrodynamic vibration standard. T. Dimoff and B. F. Payne.
Leak-resistant rotation seal for vacuum applications. F. L. Howard.
Studies on the tungsten-rhenium thermocouple to 2000 °C. D. B. Thomas.

**J. Res. NBS 67D (Radio Prop.), No. 5 (Sept.–Oct. 1963) 70 cents.**

Ionospheric VHF scattering near the magnetic equator during the International Geophysical Year. R. Cohen and K. L. Bowles.
Radio pulse propagation by a reflection process at the lower ionosphere. J. R. Johler.
Field of a horizontal magnetic dipole in the presence of a magnetoplasma halfspace. G. Tyras, A. Ishimaru, and H. M. Swarm.
Reflection of VLF radio waves from an inhomogeneous ionosphere. Part II. Perturbed exponential model. J. R. Wait and L. C. Walters.
Collisional detachment and the formation of an ionospheric *C* region. E. T. Pierce.
Magnetic torques and Coriolis effects on a magnetically suspended rotating sphere. J. C. Keith.
Radiation field characteristics of lightning discharges in the band 1 kc/s to 100 kc/s. W. L. Taylor.
Low-frequency radio propagation into a moderately rough sea. D. F. Winter.
VLF superdirective loop arrays. E. W. Seeley.
Curves of ground proximity loss for dipole antennas (a digest). L. E. Vogler and J. L. Noble.
Observations and results from the "hiss recorder" an instrument to continuously observe the VLF emissions. J. M. Watts, J. A. Koch, and R. M. Gallet.

**J. Res. NBS 67D (Radio Prop.), No. 6 (Nov.–Dec. 1963) 70 cents.**

A radiometeorological study, part I. Existing radio-meteorological parameters. J. A. Lane and B. R. Bean.
A radiometeorological study, part II. An analysis of VHF field strength variations and refractive index profiles. B. R. Bean, V. R. Frank, and J. A. Lane.
A radiometeorological study, part III. A new turbulence parameter. B. R. Bean, E. J. Dutton, J. A. Lane, and W. B. Sweezy.
Reflection of radio waves from undulating tropospheric layers. A. T. Waterman, Jr., and J. W. Strohbehn.
Oblique propagation of groundwaves across a coastline—Part I. J. R. Wait.
Oblique propagation of groundwaves across a coastline—Part II. J. R. Wait and C. M. Jackson.
On the index of refraction of air, the absorption and dispersion of centimeter waves by gases. G. Boudouris. Translated from French by G. W. Curtis.
A numerical approach to the solution of radio diffraction problems. J. K. Hargreaves and S. Hargreaves.
Inversion of radio wave absorption data to establish ionospheric properties. I. Nondeviative absorption. A. D. Wheelon.
Radiation in a lossless magneto-ionic medium at frequencies high relative to the electron gyrofrequency. J. W. Marini.
Radiation through cylindrical plasma sheaths. J. H. Harris.
Chart for determining the effects of ionospheric tilts using an idealized model. T. A. Croft and R. B. Fenwick.
Reflection of VLF radio waves from an inhomogeneous ionosphere. Part III. Exponential model with hyperbolic transition. J. R. Wait and L. C. Walters.
Families of distributions for hourly median power and instantaneous power of received radio signals. M. M. Siddiqui and G. H. Weiss.
Statistical methods in radar astronomy. Determination of surface roughness. H. S. Hayre.
Electrical conductivity of the Great Lakes. L. H. Doherty.

---

Standard X-ray diffraction powder patterns, H. E. Swanson, M. C. Morris, R. P. Stinchfield, and E. H. Evans, NBS Mono. 25, Section 2 (May 3, 1963) 35 cents.
Tensile and impact properties of selected materials from 20 to 300 °K, K. A. Warren and R. P. Reed, NBS Mono. 63 (June 28, 1963) 35 cents.
Refractive indices and densities of aqueous solutions of invert sugar, C. F. Snyder and A. T. Hattenburg, NBS Mono. 64 (June 7, 1963) 15 cents.
Reduction of data for piston gage pressure measurements, J. L. Cross, NBS Mono. 65 (June 17, 1963) 15 cents.
Tabulation of data on receiving tubes, C. P. Marsden and J. K. Moffitt, NBS Handb. 83 (May 23, 1963) $1.25. (Supersedes Handb. 68.)
Radiobiological dosimetry. Recommendations of the International Commission on radiological units and measurements, NBS Handb. 88 (April 30, 1963), Supersedes parts of Handb. 78. Handbooks 84 through 89 will completely replace H78, 25 cents.
Quarterly radio noise data, September, October, November 1962, W. Q. Crichlow, R. T. Disney, and M. A. Jenkins, NBS Tech. Note 18–16 (June 10, 1963) 60 cents.
Mean electron density variations of the quiet ionosphere, November 1959, J. W. Wright, L. R. Wescott, and D. J. Brown, NBS Tech. Note 40–9 (April 22, 1963) 35 cents.
Mean electron density variations of the quiet ionosphere, December 1959, J. W. Wright, L. R. Wescott, and D. J. Brown, NBS Tech. Note 40–10 (March 24, 1963) 35 cents.
Phototypesetting of computer output, an example using tabular data, W. R. Bozman, NBS Tech. Note 170 (June 25, 1963) 10 cents.
Curves of ground proximity loss for dipole antennas, L. E. Vogler and J. L. Noble, NBS Tech. Note 175 (May 20, 1963) 30 cents.
Table of attenuation error as a function of vane-angle error for rotary vane attenuators, W. Larson, NBS Tech. Note 177 (May 20, 1963) 75 cents.
An interpolation procedure for calculating atmospheric band absorptions from laboratory data, L. Droppleman, L. R. Megill, and R. F. Calfee, NBS Tech. Note 178 (June 3, 1963) 20 cents.
Tabulation of published data on Soviet electron devices, C. P. Marsden, NBS Tech. Note 186 (June 3, 1963) 45 cents.
Impedance of commercial Leclanché dry cells and batteries, R. J. Brodd and H. J. Dewane, NBS Tech. Note 190 (July 5, 1963) 40 cents.
Tables describing small-sample properties of the mean, median, standard deviation, and other statistics in sampling from various distributions, C. Eisenhart, L. S. Deming, and C. S. Martin, NBS Tech. Note 191 (June 14, 1963) 20 cents.
Calculations of the potential and effective diffusion constant in a polyelectrolyte solution, S. R. Coriell and J. L. Jackson, NBS Tech. Note 192 (June 28, 1963) 25 cents.
A bibliography of foreign developments in machine translation and information processing, J. L. Walkowicz, NBS Tech. Note 193 (July 10, 1963) $1.00.
National standard reference data program, background information, NBS Tech. Note 194 (June 1963) 25 cents.

## Publications in Outside Journals

Nuclear resonance and the hyperfine field in dilute alloys of nickel in iron, R. L. Streever, L. H. Bennett, R. C. La Force, and G. F. Day, J. Appl. Phys. **34**, No. 4, Pt. 2, 1050–1051 (Apr. 1963).
The history of Pt. 27, E. Wichers, Book, Temperature, Its Measurement and Control in Science and Industry **3**, Pt. 1, 259–262 (Reinhold Publ. Corp., New York, N.Y., 1962).
Some causes of resonant frequency shifts in atomic beam machines. I. Shifts due to other frequencies of excitation, J. H. Shirley,

J. Appl. Phys. 34, 783–788 (Apr. 1963).

Some causes of resonant frequency shifts in atomic beam machines. II. The effect of slow frequency modulation on the Ramsey line shape, J. H. Shirley, J. Appl. Phys. **34,** 789–791 (Apr. 1963).

On the dependence of absorption coefficients upon the area of the absorbent material, E. D. Daniel, J. Acoust. Soc. Am. **35,** No. 4, 571–573 (Apr. 1963).

The role of the International Union of pure and applied chemistry, E. Wichers, J. Chem. Doc. **3,** No. 7, 7–11 (1963).

New scale of nuclidic masses and atomic weights, E. Wichers, Nature **194,** No. 4829, 621–624 (May 19, 1962).

Note on a subgroup of the modular group, M. Newman and J. R. Smart, Proc. Am. Math. Soc. **14,** No. 1, 102–104 (Feb. 1963).

Calibration of photogrammetric lenses and cameras at the National Bureau of Standards, F. E. Washer, Photogrammetric Eng. **XXIX,** No. 1, 113–119 (Jan. 1963).

The orthobaric densities of parahydrogen, derived heats of vaporization and critical constants, H. M. Roder, D. E. Diller, L. A. Weber, and R. D. Goodwin, Cryogenics **3,** 16–22 (Mar. 1963).

Melting pressure equation for the hydrogens, R. D. Goodwin, Cryogenics **2,** No. 6, 1–3 (Dec. 1962).

Isotopic fractionation of uranium in sandstone, J. N. Rosholt, W. R. Shields, and E. L. Garner, Science **139,** 224–226 (Jan. 18, 1963).

The total electron content of the ionosphere at middle latitudes near the peak of the solar cycle, R. S. Lawrence, D. J. Posakony, O. K. Garriott, and S. C. Hall, J. Geophys. Res. **68,** 1889–1898 (Apr. 1, 1963).

Present status of our knowledge of atomic transition probabilities, W. L. Wiese, Proc. Tenth Colloquium Spectroscopic Intern., pp. 37–56 (Univ. of Maryland, College Park, Md., 1962).

Rubber and rubber products, W. P. Tyler and M. Tryon, Book, Industrial and Natural Products and Noninstrumental Methods, 6th Ed., Standard Methods of Chemical Analysis **IIB,** Chapt. 43, 2146–2226 (D. Van Nostrand Co., Inc., New York, N.Y., 1963).

The formation and oxidation of high-area carbon films, V. R. Deitz and E. F. McFarlane, Proc. Fifth Carbon Conference II, 219–232 (Pergamon Press, Inc., London, England, 1963).

Pure substance and measurement, E. Wichers, Mater. Res. Stds. **1,** No. 4, 314–315 (Apr. 1961).

The specific heat at constant volume of parahydrogen at temperatures from 15 to 90 °K and pressures to 340 atm, B. A. Younglove and D. E. Diller, Cryogenics **2,** No. 6, 1–5 (Dec. 1962).

Radiation detectors, L. Costrell, Science **139,** No. 3558, 899 (Mar. 8, 1963).

Electron spin resonance of gamma-irradiated cellulose, R. E. Florin and L. A. Wall, J. Polymer Sci. **1,** Pt. A, 1163–1173 (1963).

The speed of light, A. G. McNish, IRE Trans. Instr. **I–11,** Nos. 3 & 4, 138–148 (Dec. 1962).

Fundamentals of measurement, A. G. McNish, Electro-Technol. **53,** 113–128 (May 1963).

Optimum estimators of the parameters of negative exponential distributions from one or two order statistics, M. M. Siddiqui, Ann. Math. Stat. **34,** 117–121 (Mar. 1963).

Building a simple transistor tester, G. F. Montgomery, Electronics **36,** No. 16, 56 (Apr. 19, 1963).

Electron microscopy studies of the surfaces of magnetic recording media, F. Nesh and D. B. Ballard, IFEE Trans. Audio **AU–11,** No. 1, 15–18 (Jan. 2, 1963).

A magnetic amplifier for use with diode logic, E. W. Hogue, Proc. IEEE 1963 Intern. Conf. Nonlinear Magnetics No. T–149, 8.6–1 to 8.6–6 (Apr. 1963).

Kinetics of the acid-catalyzed hydrolysis of acetal in water-acetone solvents at 15, 25, and 35°, R. K. Wolford, J. Phys. Chem. **67,** 632–636 (1963).

Performance characteristics of split-type residential air-to-air heat pumps, J. C. Davis and P. R. Achenbach, Suppl. Bull. Inst. Intern. Refrigeration, pp. 1–7 (1961–1962).

Crystallographic changes with the substitution of aluminum for iron in dicalcium ferrite, D. K. Smith, Acta Cryst. **15,** 1146–1152 (Jan. 1963).

Methods for the analysis of rubber and related products, M. Tryon and E. Horowitz, Handb. Analytical Chemistry, Sect. 13, pp. 233–256 (McGraw-Hill Book Co., Inc., New York, N.Y., 1963).

Pressure-density-temperature relations of freezing liquid para hydrogen to 350 atmospheres, R. D. Goodwin, Cryogenics **3,** 12–15 (Mar. 1963).

Nuclear magnetic resonance in metal powders at low temperatures, R. J. Snodgrass and L. H. Bennett, J. Appl. Spectry. **17,** No. 2, 53–54 (1963).

Millimeter wavelength resonant structures, R. W. Zimmerer, M. V. Anderson, G. L. Strine, and Y. Beers, IEEE Trans. Microwave Theory Tech. **MTT–11,** 141–149 (Mar. 1963).

Lunar point-to-point communication, L. E. Vogler, Book, Technology of Lunar Exploration, Progress in Astronautics and Aeronautics **9,** 533–559 (Academic Press Inc., New York, N.Y., 1963).

Comment on "Parametric behavior of an ideal two-frequency varactor", G. F. Montgomery, Proc. IEEE **51,** No. 3, 491 (Mar. 1963).

Distribution of latitude of red arcs, E. Marovich and F. E. Roach, J. Geophys. Res. **68,** No. 7, 1885–1888 (Apr. 1, 1963).

Instability of the equatorial *F* layer after sunset, W. Calvert, J. Geophys. Res. **68,** 2591–2593 (May 1, 1963).

A simple environmental chamber for rotating beam fatigue testing machines, J. A. Bennett, Mater. Res. Std. **3,** No. 6, 480–482 (June 1963).

Standard potential of the silver-silver chloride electrode and activity coefficients of hydrochloric acid in aqueous methanol (33.4 Wt. %) with and without added sodium chloride at 25°, R. G. Bates and D. Rosenthal, J. Phys. Chem. **67,** 1088–1090 (1963).

International standardization. A new responsibility of the engineer, A. T. McPherson, Natl. Acad. Sci.-Natl. Res. Council Div. of Eng. and Ind. Res. Newsletter, No. 26, 2–4 (June 1, 1963).

Path antenna gain and comments on "properties of 400 Mc/s long-distance tropospheric circuits", W. J. Hartman, Proc. IEEE **51,** 847–848 (May 1963).

Absorption and scattering of photons by deformed nuclei, E. G. Fuller, Proc. Second All-Union Conf. Nuclear Reactions at Low and Intermediate Energies, p. 419, 1960 (Russian Academy of Science, U.S.S.R., 1962).

Microwave spectrum of tertiary butyl chloride, A. Comparison of tertiary butyl structures, D. R. Lide, Jr., and M. Jen, J. Chem. Phys. **38,** No. 7, 1504–1507 (Apr. 1, 1963).

Search for a slow component in alpha ionization, Z. Bay and R. M. Pearlstein, Phys. Rev. **130,** No. 1, 223–227 (Apr. 1963).

Evidence regarding the mechanism of fatigue from studies of environmental effects, J. A. Bennett, Acta Met. **11,** No. 7, 799–800 (July 1963).

Dielectric friction on a rotating dipole, R. Zwanzig, J. Chem. Phys. **38,** No. 7, 1605–1606 (Apr. 1, 1963).

Field-aligned *E*-region irregularities identified with acoustic plasma waves, K. L. Bowles, B. B. Balsley, and R. Cohen, J. Geophys. Res. **68,** 2485–2501 (May 1, 1963).

The association of plane-wave electron-density irregularities with the equatorial electrojet, R. Cohen and K. L. Bowles, J. Geophys. Res. **68,** 2603–2611 (May 1, 1963).

Calorimetric calibration of the electrical energy measurement in an exploding wire experiment, D. H. Tsai and J. H. Park, Exploding Wires **2,** 27–107 (Plenum Press, Inc., New York, N.Y., 1962).

Quasi-equilibrium theory of mass spectra, H. M. Rosenstock and M. Krauss, Book, Mass Spectrometry of Organic Ions, pp. 1–64 (Academic Press, Inc., New York, N.Y., 1963).

Determination of differential X-ray photon flux and total beam energy, J. S. Pruitt and H. W. Koch, Book, Nuclear Physics 5, Chapt. 2.8.2, Pt. B, 508–553, Ed. L. C. L. Yuan and C. Wu (Academic Press, Inc., New York, N.Y., 1963).

Index to the communications of the ACM, 1958–1962, W. W. Youden, Comm. ACM **6,** No. 3, 1–32 (Mar. 1963).

Electrode potentials in fused systems. VI. Membrane potentials, K. H. Stern, Phys. Chem. **67,** 893–895 (1963).

Electron spin resonance spectra of aged, γ-irradiated polystyrenes, R. E. Florin, L. A. Wall, and D. W. Brown, J. Polymer Sci. Pt. A. 1, 1521–1529 (1963).

Magnetic properties of acetinide element alloys and compounds, J. C. Eisenstein, Book, Landolt-Bornstein, 6Ed. 2, Pt. 9, p. 3.236 (Springer-Velag, Berlin, 1962).

Planar twin boundary in aluminum, T. H. Orem, Trans Met. Soc. AIME **227,** 786–788 (June 1963).

Radio noise anomalies in August 1958, C. A. Samson, J. Geophys. Res. **68,** 2719–2726 (May 1, 1963).

Optical studies of the formation and breakdown of passive films formed on iron single crystal surfaces in inorganic inhibitor solutions, J. Kruger, J. Electrochem. Soc. **110,** No. 6, 654–663 (June 1963).

The importance of environment in fatigue failure of metals, J. A. Bennett, W. L. Holshouser and H. P. Utech, Book, Fatigue of

Aircraft Structures, pp. 1–18 (Pergamon Press, Inc., London, England, 1963).

Production of Lyman alpha radiation in ion-atom collisions, G. H. Dunn, R. Geballe, and D. Pretzer, Phys. Rev. **128,** No. 5, 2200–2206 (ec. 1, 1962).

Microwave spectrum of aluminum monofluoride, D. R. Lide, Jr., J. Chem. Phys. **38,** No. 8, 2027 (Apr. 15, 1963).

Some limits on the effect of coronal self-emission upon the excitation state of coronal ions, R. N. Thomas and C. Pecker, Astrophys. J. **137,** No. 3, 967–980 (1962).

Oil baths for saturated standard cells, P. H. Lowrie, Jr., ISA J. **9,** No. 12, 47–50 (Dec. 1962).

Two-stream plasma instability as a source of irregularities in the ionosphere, D. T. Farley, Jr., Phys. Rev. Letters **10,** No. 7, 279–282 (Apr. 1, 1963).

Performance of the barium fluoride film hygrometer element on radiosonde flights, F. E. Jones, J. Geophys. Res. **68,** No. 9, 2735–2751 (May 1, 1963).

Radiation induced polymerization of propylene at high pressure, D. W. Brown and L. A. Wall, J. Phys. Chem. **67,** 1016–1019 (1963).

Dielectric friction on a moving ion, R. Zwanzig, J. Chem. Phys. **38,** No. 7, 1603–1605 (Apr. 1, 1963).

Survey of $Rb^{85}/Rb^{87}$ ratios in minerals, W. R. Shields, E. L. Garner, C. E. Hedge and S. S. Goldich, J. Geophys. Res. **68,** No. 8, 2331–2334 (Apr. 15, 1963).

Dissociation constant of pyrrolidinium ion and related thermodynamic quantities from 0 to 50°, H. B. Hetzer, R. G. Bates, and R. A. Robinson, J. Phys. Chem. **67,** 1124–1127 (1963).

Introduction to the International Symposium on Equatorial Aeronomy, R. Cohen and K. L. Bowles, J. Geophys. Res. **68,** 2359–2361 (May 1, 1963).

Interference fringes with long path difference using He-Ne laser, T. Morokuma, K. F. Nefflen, T. R. Lawrence, and T. M. Klucher, J. Opt. Soc. Am. **53,** No. 3, 394–395 (Mar. 1963).

Hydrogen retention system for pressure calibration of microphones in small couplers, W. Koidan, J. Acoust. Soc. Am. **35,** No. 4, 614 (Apr. 1963).

Numerical computation of time-dependent properties of isotopically disordered one-dimensional harmonic crystal lattices, R. J. Rubin, J. Phys. Soc. Japan **18,** Suppl. II, 63–69 (1963).

The National Bureau of Standards: Where measurement is the central theme, C. S. McCamy, Ind. Phot. **11,** No. 8, 28 (Aug. 1962).

Statistical computation of configuration and free volume of a polymer molecule with solvent interaction, J. Mazur and L. Joseph, J. Chem. Phys. **38,** No. 6, 1292–1300 (Mar. 15, 1963).

A comparison of two independent atomic time scales, J. Newman, L. Fay, and W. R. Atkinson, Proc. IEEE **51,** No. 3, 498–499 (March 1963).

Symmetry conditions for internal friction caused by jumping of point defects in crystals, J. B. Wachtman, Jr., and H. S. Peiser, J. Appl. Phys. Letter **1,** No. 1, 20–22 (Sept. 1962).

Change in occlusion of complete dentures caused by a pipe habit: a case report, J. B. Woefel and G. C. Paffenbarger, J. Am. Dental Assoc. **66,** No. 4, 478–485 (Apr. 1963).

Measurement of a maximum in the isothermal crystallization rate-temperature curve for polypentene-1, F. A. Quinn, Jr., and J. Powers, Polymer Letters (J. Polymer Sci., Pt. B) **1,** No. 7, 341–344 (July 1963).

Pressure induced shifts of infrared lines due to polar molecules, E. K. Plyler, M. A. Hirshfield, and J. H. Jaffe, J. Chem. Phys. **38,** No. 1, 257–258 (Jan. 1963).

Radio spectrum of SH, H. E. Radford and M. Linzer, Phys. Rev. Letters **10,** No. 10, 443–444 (May 15, 1963).

Electrophoretic deposits of barium titanate, V. A. Lamb and H. I. Salmon, J. Am. Ceram. Bull. **41,** No. 11, 781 (1962).

Measurement of the lattice constants of neon isotopes in the temperature range 4°–24 °K, L. H. Bolz and F. A. Mauer, Proc. 11th Annual Conf. Applications of X-ray Analysis, Denver Res. Inst. (Plenum Press, Inc., New York, N.Y., 1962).

Thermodynamic properties of gases, J. Hilsenrath, Am. Inst. Phys. Handb. 2d Ed., Sec. 4h, 101–157 (1963).

The decoration of dislocations in aluminum oxide, H. E. Bond and K. B. Harvey, J. Appl. Phys. **34,** No. 2, 440–441 (Feb, 1963).

The effect of geomagnetic crochet on cosmic ray intensity, C. S. Warwick, J. Geophys. Res. **68,** No. 10, 3303–3304 (May 15, 1963).

Design of an interferometric oil manometer for vacuum measurement, A. M. Thomas, D. P. Johnson, and J. W. Little, 1962 Trans. Ninth Natl. Vacuum Symp., Am. Vacuum Soc. pp. 468–473 (1962).

Transference numbers in pure molten sodium nitrate, R. J. Labrie and V. A. Lamb, J. Electrochem. Soc. **110,** No. 7, 810–814 (July 1963).

Unstable detonation near a hypervelocity missile, F. W. Ruegg and W. W. Dorsey, Ninth Symp. Intern. Combustion, pp. 476–477 (1963).

Vacuum-tight cylinder joints and ball-and-socket joints, L. A. Guildner and H. F. Stimson, Rev. Sci. Instr. **34,** No. 6, 658–659 (June 1963).

Exchange processes in the reaction of boron trichloride with triethylamine-boron trifluoride, T. D. Coyle, Proc. Chem. Soc. (London), p. 172 (June 1963).

Analysis of the third spectrum of praseodymium, J. Sugar, J. Opt. Soc. Am. **53,** No. 7, 831–838 (July 1963).

Study of Knudsen's method of pressure division as a means of calibrating vacuum gauges, S. Schuhmann, Trans. Ninth Natl. Vacuum Symp. Am. Vacuum Soc., pp. 463–467 (1962).

Design of low voltage electron guns, J. A. Simpson and C. E. Duyatt, Rev. Sci. Instr. **34,** No. 3, 265–268 (Mar. 1963).

Classification of two-electron excitation levels of helium, J. W. Cooper, U. Fano, and F. Prats, Phys. Rev. Letters **10,** No. 12, 518–521 (June 1963).

Photodisintegration of light nuclei, E. Hayward, Rev. Mod. Phys. **35,** No. 2, 324–331 (Apr. 1963).

Solid state physics, H. P. R. Frederikse (Editor and Part Author), Book, Am. Inst. Phys. Handbook Sec. 9, 2d Ed. (McGraw-Hill Book Co., New York, N.Y., 1963).

Very low temperature data. Properties of paramagnetic salts, R. P. Hudson, AIP Handb. 2d Ed. Sec. 5g, pp. 222–223 (1963).

VLF propagation—theory and experiment, J. R. Wait, Monograph, Electromagnetic Waves in Stratified Media, Chapt. IX, pp. 264–288 (Pergamon Press, Inc., Oxford, England, 1962).

Plastics, G. M. Kline, Americana annual, p. 535 (Americana Corp., Chicago, Ill., 1963).

Sum rules relating coherent X-ray scattering data to the diamagnetic nuclear shielding constant and to the self-energy of the charge distribution of the scatterer, J. N. Silverman and Y. Obata, J. Chem. Phys. **38,** No. 5, 1254–1255 (1963).

Present status of our knowledge of atomic transition probabilities, W. L. Wiese, Proc. Xth Colloq. Spectroscopicum Intern., pp. 37–56 (Ed. E. R. Lippincott and M. Margoshes, Spartan Books, Washington, D.C., 1963).

Collective correlation of plasma, C. M. TchenCzechoslovak J. Phys., Proc. 2d Conf. Electronics XXI, No. 5–6, 516–521 (Apr. 1962).

Double-photon photodetachment of negative ions, S. Geltman, Phys. Rev. Letters **4,** No. 3, 168–169 (Apr. 1, 1963).

Mathematical analysis of thermal environment in underground shelters, P. R. Achenbach, T. Kusuda, and F. J. J. Drapeau, ASHRAE Symp. Survival Shelters, pp. 9–33 (June 1962).

Effect of insulation on the weathering of smooth-surfaced, built-up roofs exposed to solar heating, W. C. Cullen and W. H. Appleton, Am. Roofer and Building Improvement Building Contractor Mag. Pt. I, **53,** No. 3, 19–22 (Mar. 1963); Pt. II, **53,** No. 4, 14 (Apr. 1963).

The precise evaluation of lens distortion, F. E. Washer, Photogrammetric Eng. XXXI 327–332 (Mar. 1963).

Pressure broadening as a prototype of relaxation, U. Fano, Phys. Rev. **131,** No. 1, 259–268 (July 1, 1963).

The atomic spectra of the rare earths: their presence in the sun, C. E. Moore, Appl. Opt. **2,** No. 7, 665–674 (July 1963).

The conductive-disk method of measuring the thermal conductivity of insulators, H. E. Robinson, Suppl. Bull. Intern. Inst. Refrigeration, Annexe 1962–1, 43–50 (Aug. 1962).

Electrodeposition of alloys. Principles and practice. Vol. I. General Survey, principles, and alloys of copper and of silver, and Vol. II, Practical and specific information, A. Brenner, Book, Academic Press, Inc., New York, N.Y., 1963.

Factorial designs and the direct product, B. Kurkjian and M. Zelen, Bull. Intern. Stat. Inst. **XXXIX,** Pt. 2, 509–519 (1962).

Electron affinity of atomic iodine, B. Steiner, M. L. Seman and L. M. Branscomb, J. Chem. Phys. **37,** No. 6, 1200–1202 (June 1962).

The contact properties of thin films on semiconductors, T. Higier and G. G. Harman, November 1961 Contact Symp. Proc. Univ. Maine (Mar. 20, 1963).

Temperature scales, thermocouples and resistance thermometers, H. F. Stimson, J. F. Swindells, and R. E. Wilson, Am. Inst. Phys. Handb. 2d Ed., Sec. 4a, pp. 2–21 (1963).

Field-emission study of carbon monoxide on tantalum, R. Klein and L. B. Leder, J. Chem. Phys. **38,** No. 8, 1866–1872 (Apr. 15, 1963).

Field emission observations of carbon on tantalum, R. Klein and L.

Leder, J. Chem. Phys. **38**, No. 8, 1863–1866 (Apr. 1963).

Electron impact ionization of atomic hydrogen, S. Geltman, M. R. Rudge, and M. J. Seaton, Proc. Phys. Soc. **81**, Pt. 2, No. 520, 375–377 (Feb. 1, 1963).

Electron energy levels and their relationship to lattice defects in reduced rutile, J. H. Becker and W. R. Hosler, (Proc. Intern. Conf. Crystal Lattice Defects), J. Phys. Soc. Japan **18**, Suppl. II, 152–160 (1963).

Addendum: Vibration-rotation spectra of BrCN, A. Maki, J. Chem. Phys. **38**, No. 5, 1261–1262 (May 1963).

The present status of radiometric calibrations in the ultraviolet spectrum, R. Stair, Symp. on Military Applications of Ultraviolet Radiation, Rept. No. LAS–TR–199–37 (Nov. 1962).

Acoustic properties of liquids, M. Greenspan, Am. Inst. Phys. Handb. 2d Ed. Sec. 3e, pp. 71–82 (1963).

Reflection mechanisms for sporadic *E*, K. Tao and J. R. Wait, Book, Ionospheric Sporadic *E*, Ed. E. K. Smith and S. Matsushita, Chapt. 4, Pt. II, pp. 13–27 (Pergamon Press, Inc., Oxford, England, 1962).

Gas measurement and other services of the National Bureau of Standards, C. T. Collett, Am. Gas Assoc. Preprint DMC–63, pp. 1–3 (1963).

Review of reviews of atomic spectra, W. Meggers, Appl. Opt. **2**, No. 7, 657–663 (July 1963).

Molecular parameters of ethane, W. J. Lafferty and E. K. Plyler, J. Chem Phys. **37**, No. 11, 2688–2692 (Nov. 1962).

New autioionizing atomic energy levels in He, Ne, and A, R. P. Madden and K. Codling, Phys. Rev. Letters **10**, No. 12, 516–518 (June 15, 1963).

A modification of the Born-Mayer potential function as applied to the crystalline alkali halides, T. B. Douglas, J. Chem. Phys. **38**, No. 10, 2461–2466 (May 15, 1963).

Discussion of errors of a recommended standard resistor-noise test system, N. Newman and G. T. Conrad, Jr., IRE Trans. Professional Group on Component Parts **CP–9**, No. 4, 180–192 (Dec. 1962).

Microbalance techniques for high temperature applications—further developments, N. J. Carrera and R. F. Walker, Book, Vacuum Microbalance Techniques, **3**, 153–177 (Plenum Press, Inc., New York, N.Y., 1963).

**Research in American Society for Testing Materials**, B. L. Wilson, Mater. Res. Std. **2**, No. 8, 694–695 (Aug. 1962).

Heat capacities, G. T. Furukawa and T. B. Douglas, Am. Inst. Phys. Handb. 2d Ed., Sec. 4e, pp. 47–63 (1963).

High resolution spectra in the region from 2 to 6$\mu$, E. K. Plyler and E. D. Tidwell, (Proc. Intern. Molecular Conf.), Book, Advances in Molecular Spectroscopy, pp. 1336–1342 (Pergamon Press Inc., New York, N.Y., 1962).

Impressions of Japan, R. S. Marvin, Japanese J. Kobunshi (High Polymers), Soc. High Polymers **12**, No. 130, 53 (1963).

Theory of the electronic polarizabilities of ions in crystals: Application to the alkali halide crystals, A. R. Ruffa, Phys. Rev. **130**, No. 4, 1412–1423 (May 15, 1963).

Annual report for IAU commission 14, C. E. Moore, Trans. Astronomical Union XIA97–XIB, 208 (1962).

Atomic flame reactions involving N-atoms, H-atoms and ozone, D. Garvin and H. P. Broida, 9th Intern. Combustion Symp., pp. 678–688 (1963).

The effect of moisture on heat transfer through insulated flat-roof constructions, F. J. Powell, Suppl. Bull. Intern. Inst. Refrigeration, Annexe 1962–1, 61–70 (Aug. 1962).

Infrared spectrum of acetylene-$d_1$, W. J. Lafferty, E. K. Plyler, and E. D. Tidwell, J. Chem. Phys. **37**, No. 9, 1981–1988 (Nov. 1962).

Nickel and alloys, M. R. Meyerson, Encyclopedia of Engineering Materials and Processes, pp. 439–442 (Reinhold Publ. Corp., New York, N.Y., 1963).

Audiofrequency dispersion effects in lanthanide salts at low temperatures, R. P. Hudson and B. W. Mangum, Book, Magnetic and Electric and Electric Resonance and Relaxation, Ed. J. Smidt, pp. 135–146 (North-Holland Publ. Co., Amsterdam, The Neitherlands, 1963).

Some problems in temperature measurements from line spectra, J. T. Jefferies, Book, Temperature, Its Measurement and Control in Science and Industry **3**, No. 1, 703–711 (1962).

Nonreciprocity of propagation of VLF radio waves along the magnetic equator, D. D. Crombie, Proc. IEEE **51**, No. 4, 617–618 (Apr. 1963).

Infrared spectrum of carbon tetrafluoride, A. Maki, E. K. Plyler, and R. Thibault, J. Chem. Phys. **37**, No. 8, 1899–1900 (Oct. 1962).

Matrix effects in the gaseous H atom—condensed olefin system; surface reaction—olefin diffusion model, R. Klein and M. D. Scheer, J. Phys. Chem. **66**, No. 12, 2677–2681 (Dec. 1962).

A Note on domain conversions of $BaTiO_3$, P. H. Fang and W. S. Brower, J. Appl. Phys. **34**, No. 5, 1516 (May 1963).

Stability relations of dravite, a tourmaline, C. R. Robbins and H. S. Yoder, Jr., Geophysical Laboratory Yearbook 1961–62 (Note) (Carnegie Institute of Washington, Feb. 1963).

An alignment interferometer, J. B. Saunders, Appl. Opt. **2**, No. 5, 541 (May 1963).

An application of the peek-a-boo principle to information retrieval, J. Stern, Proc. Symp. Materials Information Retrieval, pp. 93–115 (Nov. 28–29, 1962).

Critical review of sugar color and turbidity measurements, F. G. Carpenter and V. R. Deitz, Sugar Ind. Tech. **21**, Paper 65, 32–52 (1962).

Molecular vib-rotors. The theory and interpretation of high resolution infrared spectra, H. C. Allen, Jr., and P. C. Cross, Book, John Wiley & Sons, Inc., New York, N.Y., 1963.

On the vacuum ultraviolet refectance of evaporated aluminum before and during oxidation, R. P. Madden, L. R. Canfield, and G. Haas, J. Opt. Soc. Amm. **53**, No. 5, 620–625 (May 1963).

An LCAO–MO–SCF study of the structure of $HO_2$, M. E. Boyd, Program Comm. Intern. Symp. on Molecular Structure and Spectroscopy, pp. B404–1–B404–4 (Science Council of Japan, Tokyo, 1962).

Burning behavior of building finish materials: Two-test methods, A. R. Robertson, Spec. Publ. Fed. Fire Council, pp. 21–35 (Apr. 1962).

The kinetics of description II: $Ca^+$ and $Ba^+$ from rhenium, M. D. Scheer and J. Fine, J. Chem. Phys. **38**, No. 2, 307 (Jan. 1963).

A computer for the Abel inversion, C. R. Yokley and J. B. Shumaker, Rev. Sci. Instr. **34**, No. 5, 551–557 (May 1963).

Rotation-vibration constants of acetylene, E. K. Plyler, E. D. Tidwell, and T. A. Wiggins, J. Opt. Soc. Am. **53**, No. 5, 589–593 (May 1963).

Calculation of crystallite size distributions from X-ray line broadening, A. Bienenstock, J. Appl. Phys. **34**, No. 5, 1391 (May 1963).

Osmotic and activity coefficients of tetraethylammonium iodide in aqueous solution at 25°, V. E. Bower and R. A. Robinson, Trans. Faraday Soc. **59**, No. 488, 1717–1719 (Aug. 1963).

Dependence of the electrical conductivity and thermoelectric power of pure and aluminum-doped rutile on equilibrium oxygen pressure and temperature, J. Yahia, Phys. Rev. **130**, No. 5, 1711–1719 (June 1, 1963).

A portable impedance tube, R. D. Berendt and H. A. Schmidt, Jr., J. Acoust. Soc. Am. **35**, No. 7, 1049–1052 (July 1963).

Inertial seismograph design—limitations in principle and practice, B. S. Melton and D. P. Johnson, Proc. IRE **50**, No. 11, 2328–2339 (Nov. 1962).

Nuclear moment of $Ni^{61}$ from nuclear resonance studies in steady external magnetic fields, R. L. Streever, Phys. Rev. **128**, No. 4, 1632–1633 (Nov. 15, 1962).

Electromagnetic waves in stratified media, J. R. Wait, Book, Pergamon Press, Inc., New York, N.Y., 1962.

Absolute magnetic susceptibilities by the Gouy and the Thorpe-Senftle methods, G. A. Candela and R. E. Mundy, IRE Trans. Instr. **I–11**, No. 3–4, 106–109 (Dec. 1962).

The following papers were published in the Proc. Intern. Conf. Ionosphere, London, 1962 (The Institute of Physics and The Physical Society, London, England):

A model of the atmosphere and the ionosphere in the *E* and *F*1 regions, R. B. Norton, T. E. VanZandt, and J. S. Denison, pp. 26–34.

Doppler studies of the ionospheric effects of solar flares, K. Davies, pp. 76–83.

Ionospheric variations during geomagnetic storms, S. Matsushita, pp. 120–127.

The location of the irregularities responsible for ionospheric scintillation of a radio source, H. J. A. Chivers, pp. 258–266.

Equatorial spread-*F* motions, W. Calvert, K. Davies, E. Stiltner, and J. T. Brown, pp. 316–322.

Evidence for field-aligned ionization irregularities between 400 and 1000 km above the earth's surface, W. Calvert, T. E. VanZandt, R. W. Knecht, and G. B. Goe, pp. 324–329.

Experimental observations and theoretical calculations leading to a model for the lower ionosphere, R. H. Doherty, pp. 428–434.

Radio wave reflections at a continuously stratified plasma with electron collision frequency proportional to energy and arbitrary magnetic induction, J. R. Johler, pp. 436–445.

Very low frequency propagation in the earth-ionosphere waveguide of non-uniform width, J. R. Wait, pp. 446–451.

**Publications for which a price is indicated are available by purchase from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 20402 (foreign postage, one-fourth additional). Reprints from outside journals and the NBS Journal of Research may often be obtained directly from the authors.*

# INDEX TO VOLUME 67B

## January to December 1963


### A

Page

Algorithm for obtaining an orthogonal set of individual degrees of freedom for error ..... 19
An analysis of pedestrian queueing ..... 229
Artificial satellite, zonal harmonic perturbations of an accurate reference orbit ..... 191

### B

*Bender, B. K., A. J. Goldman*, Maximum cellular Boolean functions and perfect Gray codes ..... 77
Bessel functions of the first kind, $J_n'(x)$, $21 \leq n \leq 51$, $0 \leq x \leq 100$, zeros of the first derivatives ..... 181
Betti's reciprocal theorem, meaning of ..... 85
Bibliography, selected, of statistical literature: supplement, 1958–1960 ..... 91
Boolean functions and cellular perfect Gray codes ..... 77
*Börsch-Supan, W., H. Oser*, Numerical computation of the temporal development of currents in a gas discharge tube ..... 41
*Bramble, J. H., L. E. Payne*, Effect of error in measurement of elastic constants on the solutions of problems in classical elasticity ..... 157
—, —, Error bounds in the pointwise approximation of solutions of elastic plate problems ..... 145

### C

*Cameron, J. M.*, An algorithm for obtaining orthogonal set of individual degrees of freedom for error ..... 19
Configuration, $f^5$, eigenfunctions ..... 169
Consistent ordering, Boolean process formulations ..... 249
Currents in a gas discharge tube, numerical computations of the temporal development ..... 41

### D

*Deming, Lola S.*, Selected bibliography of statistical literature: supplement, 1958–1960 ..... 91
Derivatives, first, of Bessel functions of the first kind, $J_n'(x)$, $21 \leq x \leq 51$, $0 \leq x \leq 100$ zeros ..... 181

### E

Effect of error in measurement of elastic constants on the solutions of problems in classical elasticity ..... 157
Effect of molecular weight on viscoelastic properties of polymers as predicted by a molecular theory ..... 87
Eigenfunctions of the $f^5$ configuration ..... 169
*Eisenstein, J. C.*, Eigenfunctions of the $f^5$ configuration ..... 169
Elastic constants, effect of error in measurement, on the solutions of problems in classical elasticity ..... 157
Elastic plate problems, error bounds in the pointwise approximation of solutions ..... 145
*Epstein, L. F., J. H. Hubbell*, Evaluation of a generalized elliptic integral ..... 1
Error bounds in the pointwise approximation of solutions of elastic plate problems ..... 145
Evaluation of a generalized elliptic integral ..... 1

### F

*Fell, H. M. Newman, E. Ordman*, Tables of genera of groups of linear fractional transformations ..... 61

### G

Games, completely mixed, recognition ..... 23
Gas discharge tube, temporal development of currents, numerical computation ..... 41
*Goldman, A. J.*, Recognition of completely mixed games ..... 23
—, *B. K. Bender*, Maximum cellular Boolean functions and perfect Gray codes ..... 77
Graphs of finite idempotent Boolean relation matrices ..... 249
Gray codes, perfect, maximum cellular Boolean functions ..... 77
Groups of linear fractional transformations, tables of genera ..... 61

### H

Harmonic perturbations, zonal, of an accurate reference orbit of an artificial satellite ..... 191
***Hubbell, J. H., L. F. Epstein*, Evaluation of a generalized elliptic-type integral** ..... 1
Hypo-elasticity, remarks ..... 141

### I

Idempotent Boolean relation matrices ..... 249
Inductor, computable, a new type ..... 31
Information systems, mathematical models ..... 249
Inspection programs for randomly failing equipment, optimal periodic ..... 223
**Integral, generalized elliptic-type, evaluation** ..... 1

### K

*Kearsley, E. A.*, Solutions of the equation $\Psi_{zz} + \frac{1}{x}\Psi_z + Kx^n e^{\Psi} = 0$ ..... 245

### M

*Marvin, R. S., H. Oser*, Effect of molecular weight on viscoelastic properties of polymers as predicted by a molecular theory ..... 87
Maximum cellular Boolean functions and perfect Gray codes ..... 77
Measurement of elastic constants on the solutions of problems in classical elasticity ..... 157
Molecular weight, effect on viscoelastic properties of polymers as predicted by a molecular theory ..... 87
*Morgenthaler, G. W., H. Reismann*, Zeros of the first derivatives of Bessel functions of the first kind, $J_n'(x)$, $21 \leq n \leq 51$, $0 \leq x \leq 100$ ..... 181

### N

New type of computable inductor ..... 31
*Newman, M., H. Fell, E. Ordman*, Tables of genera of groups of linear fractional transformations ..... 61
Numerical computations of the temporal development of current in a gas discharge tube ..... 41

### O

Optimal periodic inspection programs for randomly failing equipment ..... 223
Orbit, accurate reference, of an artificial satellite, zonal harmonic perturbations ..... 191
*Ordman, E., H. Fell, M. Newman*, Tables of genera of groups of linear fractional transformations ..... 61
Orthogonal set of individual degrees of freedom for error, an algorithm for obtaining ..... 19
*Oser, H., W. Börsch-Supan*, Numerical computation of the temporal development of currents in a gas discharge tube ..... 41
—, *R. S. Marvin*, Effect of molecular weight on viscoelastic properties of polymers as predicted by a molecular theory ..... 87

### P

*Page, C. H.*, A new type of computable inductor ..... 31
*Payne, L. E., J. H. Bramble*, Effect of error in measurement of elastic constants on the solutions of problems in classical elasticity ..... 157
—, —, Error bounds in the pointwise approximation of solutions of elastic plate problems ..... 145
Pedestrian queueing, an analysis ..... 229
Pointwise approximation, error bounds in, of solutions of elastic plate problems ..... 145
Polymers, viscoelastic properties of, effect of molecular weight on, as predicted by a molecular theory ..... 87

### Q

Queueing, pedestrian, an analysis ..... 229

### R

Recognition of completely mixed games ..... 23
*Reismann, H., G. W. Morgenthaler*, Zeros of first derivatives of Bessel functions of the first kind, $J_n'(x)$, $21 \leq n \leq 51$, $0 \leq x \leq 100$ ..... 181
Reliability function, equipment ..... 223
Remarks on hypo-elasticity ..... 141
*Rosenblatt, D.*, On the graphs of finite idempotent Boolean relation matrices ..... 249

### S

Satellite, artificial, zonal harmonic perturbations of an accurate reference orbit ..... 191
Selected bibliography of statistical literature: supplement, 1958–1960 ..... 91
Solutions of elastic plate problems, error bounds in the pointwise approximation ..... 145
Solutions of problems in classical elasticity, effect of error in measurement of elastic constants ..... 157
Solutions of the equation $\Psi_{zz} + \frac{1}{x}\Psi_z + K_x{}^n e^{\Psi} = 0$ ..... 245

### T

Tables of genera of groups of linear fractional transformation ..... 61
The meaning of Betti's reciprocal theorem ..... 85
Traffic, pedestrian queueing ..... 229
*Truesdell, C.*, Remarks on hypo-elasticity ..... 141
—, The meaning of Betti's reciprocal theorem ..... 85

### V

*Vinti, J. P.*, Zonal harmonic perturbations of an accurate reference orbit of an artificial satellite ..... 191
Viscoelastic properties of polymers as predicted by a molecular theory, effect of molecular weight ..... 87

### W

*Weiss, G. H.*, An analysis of pedestrian queueing ..... 229
—, Optimal periodic inspection programs for randomly failing equipment ..... 223

### Z

Zeros of first derivatives of Bessel functions of the first kind, $J_n'(x)$, $21 \leq n \leq 51$, $0 \leq x \leq 100$ ..... 181
Zonal harmonic perturbations of an accurate reference orbit of an artificial satellite ..... 191




U.S. GOVERNMENT PRINTING OFFICE : 1963 OL—699-888

All of the equations and text of this publication were composed on the keyboard of the Linofilm Photocomposition System, eliminating all hand insertion of equation characters.

UNIVERSITY MICROFILMS, INC.
ANN ARBOR, MICHIGAN

UNITED STATES
GOVERNMENT PRINTING OFFICE
DIVISION OF PUBLIC DOCUMENTS
WASHINGTON, D.C. 20402

OFFICIAL BUSINESS

PENALTY FOR PRIVATE USE TO AVOID PAYMENT OF POSTAGE, $300 (GPO)

JOURNAL OF RESEARCH of the National Bureau of Standards Vol. 67B, No. 4 October–December 1963
B. Mathematics and Mathematical Physics

## Contents

| | Page |
|---|---|
| Zonal harmonic perturbations of an accurate reference orbit of an artificial satellite. John P. Vinti. | 191 |
| Optimal periodic inspection programs for randomly failing equipment. George H. Weiss. | 223 |
| An analysis of pedestrian queueing. George H. Weiss. | 229 |
| Solutions of the equation $\Psi_{xx}+\frac{1}{8}\Psi_x+Kx^n e^{\Psi}=0$. Elliot A. Kearsley. | 245 |
| On the graphs of finite idempotent Boolean relation matrices. David Rosenblatt. | 249 |
| Publications of the National Bureau of Standards. | 257 |
| Index to volume 67B. Mathematics and Mathematical Physics, January to December 1963. | 265 |

**Library of Congress Catalog Card Number: 62–4414**

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 20402. Price 75 cents (single copy). Subscription price: $2.25 a year, $0.50 additional for foreign mailing.

UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1963